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Report No. 226

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CONTRIBUTION OF APOLLO LUNAR PHOTOGRAPHY TO THE ESTABLISHMENT OF SELENODETIC CONTROL

by
Athanasios Dermanis

Prepared for the
National Aeronautics and Space Administration
Johnson Space Center
Houston, Texas 77058

Final Report
Contract No. NAS 9-13093
OSURF Project No. 3487-A1

The Ohio State University
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PREFACE

This project was under the supervision of Professor Ivan I. Mueller, Department of Geodetic Science at The Ohio State University, Columbus, and under the technical direction of Mr. Richard L. Nance, Code TF541 Mapping Sciences Branch, Earth Observation Division, NASA/JSC, Houston Texas. The contract was administered by the Facility and Laboratory Support Branch, Code BB 631/B4 NASA/JSC, Houston, Texas.

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1. INTRODUCTION

Among the various types of available data relevant to the establishment of geometric control on the moon, the only one covering significant portions of the lunar surface (20 %) with sufficient information content, is lunar photography, taken at the proximity of the moon from lunar orbiters. The naturally suited tool for the analysis of such data is the well known method of analytical phototriangulation. However, analytical phototriangulation in its earth-bound applications has not been traditionally viewed by photogrammetrists as a method to establish, but rather to extend or densify already existing geodetic control. The lack of high quality geometric control on the moon requires the introduction of orbital control which can be established from earth-based tracking of lunar orbiters using models containing information on the lunar ephemeris, on the rotations of the earth and the moon, station positions on the earth and on the moon and on the lunar gravity field.

Past experience from lunar phototriangulation with separately determined and constrained orbital control shows unacceptable inconsistencies [7], and there has been a call for the employment of advanced orbit determination techniques emphasizing the geometric aspects rather than the previously employed gravimetric aspects. Such techniques have become available mainly through the philosophy expressed in the work of Ingram and Tapley [9].

This work attempts to explore certain new ideas aimed in somewhat different directions. The systematic pattern in some of the inconsistencies mentioned above [7, Fig. 5 and 6] strongly indicates that they are at least partly due to referencing the orbit geometry to poorly determined frames, e.g., to the principal moments of inertia axes or to the selenographic systems.

In this paper the idea of "free geodetic networks" [4] is introduced as a tool for the statistical comparison of the geometric aspects of the various data used. Further, methods are developed for the updating of the statistics of observations and the a priori parameter estimates to obtain statistically consistent solutions by means of the "optimum relative weighting" concept.

2. REVIEW OF AVAILABLE DATA

Although other types of data such as laser ranging to the moon and differential VLBI are relevant to the establishment of selenodetic control, the emphasis here is on the utilization of orbital metric photography in conjunction with supporting laser altimetry and stellar photography and of a priori estimates of satellite positions. More specifically, the data here considered have been obtained by the Mapping Camera System aboard the spacecraft during the Apollo 15, 16 and 17 missions. Since a detailed description of the system can be found in [13] and [3], only a general description is given here.

The Mapping Camera System is composed of a terrain camera, an associated stellar camera and a laser altimeter. The terrain camera has a focal length of 76 mm, a 115 mm x 115 mm format and reseau marks with a spacing of 10 mm recorded on every photograph for film deformation compensation. The camera is capable of compensating for forward motion and the shutter interval is automatically set by means of separate sensor measuring the brightness of the lunar surface. Resolution depends on the solar altitude with an optimum of 90 lines per mm. The stellar camera has a 76 mm focal length, a format of 32 mm x 25 mm and a reseau grid with 5 mm spacing. The laser altimeter records the spacecraft altitude with an accuracy of ± 2 m.

During each revolution as the spacecraft passes over the sun-illuminated part of the lunar surface, the terrain and stellar cameras and the altimeter operate simultaneously to provide three types of observations at each exposure time. The terrain camera takes a strip of almost vertical lunar photographs, while the stellar camera takes a corresponding series of star photographs in the direction of the flight about 4° to 6° above the horizon. At the same time the altimeter measures the distance between the camera exposure point and the intersection of the camera axis with the lunar surface. The terrain camera also records this laser illuminated point. From the star field photographs, the orientation of the stellar camera is determined with respect to a "star catalog" system and

the orientation of the terrain camera in the same system can be determined using the relative orientation of the two cameras known from preflight calibrations.

Earth-based tracking of the spacecraft is used to determine the orbit geometry, more specifically, the coordinates of the camera with respect to some moon-fixed coordinate system at each exposure time.

3. COMMENTS ON COORDINATE SYSTEMS

3.1 Coordinate Systems in General

The concept of geodetic control traditionally has always been connected to the use of coordinates with respect to some frame of reference. Although coordinates are not the only necessary means to represent geodetic control, they have been introduced in practice as a matter of convenience because they allow the use of Cartesian analytic geometry, inspite of possible pitfalls. Reference frames can be divided into natural and conventional frames. The former naturally arise from the physics of a given situation (e.g., principal axes of inertia), while the latter are arbitrarily set to meet some criteria. Coordinates have a physical meaning and they can be determined (estimated) from observations only when they are referred to the natural frames of reference.

While some observations are invariant under coordinate system transformations and therefore they can be analyzed in any arbitrary reference frame, other observations are connected with certain reference frames and must be analyzed therein. For example, while range or range rate observations are coordinate system invariant, the stellar camera observations are naturally variant with respect to the "star catalog" frame of reference.

Modeling the physical processes is also connected to certain reference frames. For example, lunar theory can be viewed as the time history of the geocenter to the lunar center of mass vector, and the physical librations of the moon are connected to the principal axes of inertia selenocentric reference frame. For the foregoing reasons it seems advisable to define precisely the reference frames to be used in this report.

3.2 Traditional Coordinate Systems

The first reference frames for the description of coordinates of lunar features

emerged at times when the only possible means of observations were the low accuracy earth-based optical observations, such as heliometry and later lunar photography. In view of the low accuracy of these observations, some aspects in the precise definition of coordinate systems may have seemed to be too theoretical and/or irrelevant. As a result, errors are frequent in selenographic papers and the confusion found its way even into astronomical almanacs [6]. The main problem arises from the failure to distinguish between the following four directions (supposedly) through the selenocenter:

1) The direction of the axis of maximum moment of inertia.
2) The direction of the instantaneous axis of lunar rotation.
3) The Cassini axis, i.e., the axis whose change of orientation with respect to an inertial (ecliptic) system is governed by the three empirical laws of Cassini.
4) The direction of the mean lunar rotation axis.

A detailed description of the current and past situation is given in a paper by Habibullen (1971) [6]. His terminology is adopted also for this paper. The following four selenocentric systems are considered:

1) True Selenographic System — The z axis is the instantaneous rotation axis of the moon.

2) Dynamical System — This one is usually referred to as the principal axes of inertia system. It is defined as the Cartesian selenocentric system for which the following products of inertia vanish:

$$D = \int_M yz\, dm = 0, \quad E = \int_M xz\, dm = 0, \quad F = \int xy\, dm = 0.$$

3) Cassini Selenographic System — This system is such that its rotation with respect to the ecliptic system is given at any epoch by three Eulerian angles defined as follows:

$$\psi_0 = \Omega$$
$$\omega_0 = 180^\circ + (\ell_☽ + \Omega)$$
$$\theta_0 = I$$

where Ω is the mean longitude of the ascending node of the lunar orbit; $\ell_☽$ the mean lunar longitude and I the inclination of the Cassini equator (not the orbital

plane) with respect to the ecliptic.

4) Mean Selenographic System — The z axis is the mean rotation of the moon.

For systems (2) and (3) the direction of the x axis is defined. For systems (1) and (4) the "zero point" on the corresponding equator remains to be defined or to be arbitrarily selected. The only moon-fixed coordinate system by definition is the Dynamical System. Systems (1) and (3) can also be considered moon-fixed if they are referred to a certain reference epoch. System (4) has the most complicated definition and though it is the most widely used, it is the least well realized (established). Formally, a mean position is to be defined by means of a time average of the time variant positions, i.e., it involves an integral with respect to time over the interval $(-\infty, +\infty)$. Such an integration could be carried out only if the variation of the position of the instantaneous axis of rotation with respect to the rigid lunar body has a purely periodic character. Since this is not necessarily the case, the integration has to be realized over a certain time interval and then system (4) becomes moon-fixed with respect to this certain fixed time interval.

The Dynamical (coordinate) System is related to the Cassini Selenographic System through the relations [6]:

$$
\begin{aligned}
\psi &= \psi_0 + \sigma \\
\varphi &= \varphi_0 + (\tau - \sigma) \\
\theta &= \theta_0 + \rho
\end{aligned}
$$

where σ, ρ and τ are the physical librations in node, inclination and longitude, respectively. The difficulty in detecting the position of the principal axes of inertia by means of present observations necessitates the use of an "observable" system such as the Mean Selenographic or the True Selenographic Systems referred to some fixed epoch. In practice, therefore, the physical libration parameters σ, ρ and τ connect the Cassini System to one of the above systems.

In earliest works the mean rotation axis of the moon (Mean Selenographic System) was taken as identical to the principal axis of maximum inertia (Dynamic System) [6]. The distinction between these systems is not critical when constructing a libration theory in which the product of inertia D, E, F are to be constrained

to zero, because the products of inertia in the Mean Selenographic System are close enough to zero. The detection of the directions of the principal axes of inertia with respect to another moon-fixed system is possible only through the knowledge of the gravity field, specifically the second degree harmonics, with respect to the moon-fixed system. In Appendix C the possibility and methods of determining these directions are dealt with separately.

3.3 Natural and Arbitrary Coordinate Systems

From the point of view of establishing geometric control, the primary objective is to determine the shape and scale of a network formed by a cluster of points on the lunar surface. The shape and scale of a network is determined when the following quantities can be determined:

1) The angle $P_i \hat{P}_j P_k$ between lines $P_i P_j$, $P_j P_k$ through any points P_i , P_j , P_k in the network.

2) The distance d_{ij} of the line segment $P_i P_j$ joining any two points P_i and P_j in the network.

Group (1) quantities determine only the shape of the network.

Alternatively, for representing the network geometry one may use the coordinates of the points with respect to a coordinate system which is invariant in space with respect to the rigid network of points. In this case both the angles $P_i \hat{P}_j P_k$ and the distances d_{ij} are functions of the Cartesian coordinates of the points, which functions are invariant with respect to the choice of a particular coordinate system. This reflects the fact that the shape and scale of the network is independent of the choice of the reference system.

The coordinates of the points with respect to some particular reference frame contain more information than specified under shape and scale. They also define the position of the coordinate system with respect to the network of points. If the coordinate system is a natural one, i.e., if it is the consequence of the natural characteristics of the physical objects involved in the model, and the problem is to determine the position of a network of points with respect to this system, one should proceed in two steps, as follows:

1) Find the shape and scale of the network of points.

2) Find the position of the natural system of reference with respect to the network.

A minimum number of angles and distances which uniquely determine the shape and scale of the network is called a "fundamental set." All other angles and distances can be determined as functions of the fundamental set. Such a fundamental set also serves as a representation of the shape and scale. If coordinates are to be used as a representation of the shape and scale only, the reference frame's position with respect to the network of points must be known by definition. Such a reference frame can be established by means of the concept of minimal constaints on the coordinates. A set of coordinate conditions which specify the positions of the reference frame with respect to the network is called "minimal constraints." An easy way to set such a constraint is to assign constant values to six coordinates distributed ober at least three different points. For a more detailed discussion on minimal constraints see [4], [2] and [14].

3.4 Role of Coordinates in Selenodesy

The problem of finding the position of a network of points with respect to a natural system can now be viewed in two parts:

1) Find the coordinates of the points in the network with respect to an arbitrary reference system introduced through the use of a set of minimal constraints.

2) Find a set of transformation parameters which specifies the relative positions of the natural coordinate system with respect to the arbitrary system. The natural choice of such parameters is of course, three translations and three orientation angles.

The reason for the above dichotomy becomes obvious if one brings to mind that beyond the deterministic definitions concerning positions of points, one in practice has to extract estimates of these positions from observations which can be either coordinate system dependent or not. Furthermore,

observations may depend on more than one reference system. The situation becomes even more complicated when the relative position of these systems varies with time. For example, the representation of the geometry of a system of points on the moon requires a moon-fixed system of reference. Such a system can be either arbitrary (defined by means of a set of minimal constraints), or natural. The only natural system qualifying is the Dynamical System. Its advantage is that its position depends only on distribution of masses and that it is time invariant. Lunar photography and laser altimetry provide us with data which are independent of coordinate systems and, therefore, their analysis could be performed in an arbitrary system. However, because stellar photography is star catalog system dependent and the libration theory transforms it to a moon-fixed system, the lunar photography and laser data could also be analyzed in this system. Estimates of orbital geometry are given in the form of camera exposure station positions with respect to some moon-fixed system, not necessarily the same used in the libration theory for the transformation of stellar camera observations.

For the purpose of estimating only the shape and scale of the network it is quite easy to "free" the observational data from their dependence on coordinate systems. The problem, however, of relating the network to any of the data reference frames still remains.

3.5 Coordinate Systems in the External Information Used

At this point it is appropriate to say a word about the theories of lunar motion and physical libration. A lunar theory is a solution to the problem of the variation of the position (geocenter-selenocenter) vector with time. As such, it is an integral in one form or another of the differential equations governing the motion of this vector. The solution contains some integration constants which are either the values of the initial state of the Dynamical System (numerical integration) or they are implicit functions of the initial state (analytical theories). A libration theory is similarly a solution to the problem of the rotation of the moon (or more precisely, of a moon-fixed system, usually the Dynamical System) about the selenocenter.

In both theories in addition to the problem of determining the positions vector or the set of libration angles at any epoch, in terms of the integration constants, the values of the integration constants themselves also have to be determined. This is possible only through an estimation procedure based upon actual observations. The observations are of a geometric nature, distances and angles related to points on the earth and on the moon. The position of these points is represented through their coordinates with respect to an earth-fixed or moon-fixed system, and a theory (model) must be available for the description of the relative motion of the two systems. This model can be separated into three components all with respect to an inertial system: rotation of a geocentric earth-fixed system, motion of the geocenter-selenocenter vector and the rotation of the moon-fixed selenocentric system. The choice of the geocenter and selenocenter as system origins is natural because lunar theory is developed upon differential equations governing the motion of those two centers of mass.

The choice of the Dynamical System among all possible selenocentric moon-fixed system is attractive to libration theorists [12] because of the relatively simple form that Euler's dynamical differential equations take when this system is used.

In the process of estimating the integration constants of a lunar or libration theory, the coordinates of observational or observed points can be estimated as a by-product. Their accuracy will reflect how well the position of the coordinate system with respect to these points is known.

The user of lunar ephemeris or libration theory must realize that the position of the selenocenter and the orientation of the principal moments of inertia axes given in those theories are only estimates of the position of the true selenocenter and the true orientation of the axes. The accuracy of those estimates is supposedly given by means of the associated statistics (variances and covariances). These statistics, however, are realistic only when the model is perfect and there are no computational (round-off) errors and the observational input statistics are true.

Another problem arises from the fact that before a solution is attempted,

there is a need for two sources of a priori information:

1) A libration theory for the reduction of stellar camera observations from an inertial ("star catalog") system to a moon-fixed system.

2) Orbital geometry estimates for the strengthening of the photogrammetric solution.

In general such information is available from external sources and it may or may not fit the model. In this case either the model needs to be changed (but not if it is believed to be correct), or the statistics included with the information need to be modified in some appropriate way.

4. APPROXIMATE ERROR ANALYSIS OF THE CONTRIBUTION OF THE VARIOUS DATA TYPES

The purpose of this section is to give a projection on how the different types of data contribute to the determination of geometric control on the moon. The analysis will be with respect to a strep of photographs covering an arc of approximately 180° on the lunar surface. The analysis requires four systems of reference:

System (1)

An arbitrary system introduced by the following set of minimal constraints:

$$X_1 = Y_1 = Z_1 = \varkappa_1 = \varphi_1 = \omega_1 = 0$$

where X_1, Y_1, Z_1 are the coordinates of the first camera exposure station and $\varkappa_1$, φ_1, ω_1 are the camera orientation angles in the traditional photogrammetric sense (X axis positive in the direction of the flight, Y perpendicular to the orbital plane and Z in the direction of the selenocentric vector pointing away from the moon, forming a right handed system).

System (2)

The moon-fixed system whose orientation with respect to the ecliptic system is given by the physical libration parameters to be used in the transformation of stellar camera observations (Dynamical System).

System (3)

The moon-fixed system in which estimates of camera exposure station positions are given from a previous orbit analysis.

System (4)

A special cylindric type system designed to match the geometry of a photo strip. In this system after a point is orthogonally projected into the orbital plane, its X coordinate is the angle between the projection's selenocentric vector and the Z axis of system (1). The Z axis is along

Assuming that the only source of errors is the error δp in the measurements of photo coordinates from Figure 2, it is clear that the error in direction X is

$$\delta X = 2 \frac{H}{f} \delta p . \tag{4.1}$$

Similarly, from the YZ plane

$$\delta Y = 2 \frac{H}{f} \delta p. \tag{4.2}$$

From Figure 3 the error components from the same displacements δp in the direction Z are

$$\frac{\delta Z_1}{\delta p \frac{H}{f}} = \frac{H}{B_1}$$

and

$$\frac{\delta Z_2}{\delta p \frac{H}{f}} = \frac{H}{B_2} .$$

Assuming that $B_1 = B_2 = \frac{B}{2}$, the total error is

$$\delta Z = \delta Z_1 + \delta Z_2$$

or

$$\delta Z = \frac{4}{B} \frac{H^2}{f} \delta p. \tag{4.3}$$

From Figure 4 the following obvious relationships can be derived:

$$\sqrt{H^2 + B^2}\, \delta \varphi_1 = \Delta X \sin \theta$$

$$\sin \theta = \frac{H}{\sqrt{H^2 + B^2}}$$

$$\Delta X = \frac{H}{f} \delta p .$$

Substituting the last two equations into the first one yields

$$\delta \varphi_1 = \frac{H^2}{H^2 + B^2} \frac{1}{f} \delta p ,$$

the selenocenter vector of the projection and the coordinate Z is the distance from the orbit. (See Figure 1).

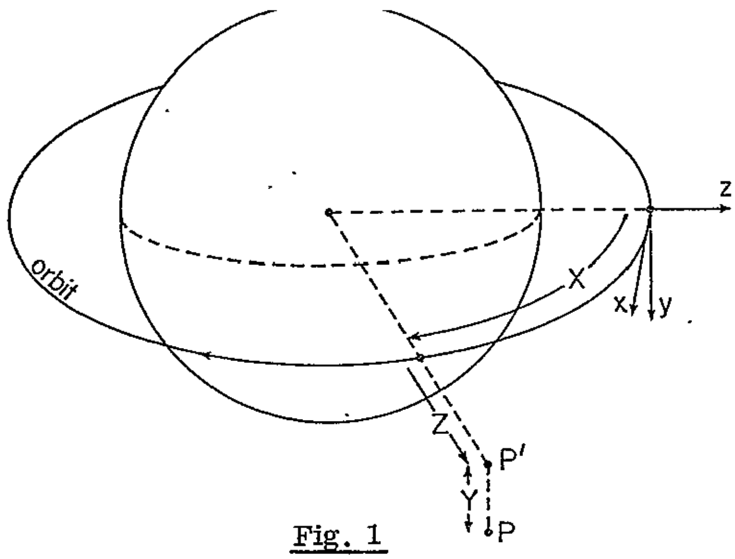


Fig. 1

4.1 Errors in the Relative Orientation of Two Cameras

Figure 2 illustrates in the XZ plane of system (4), the geometry of two vertical photographs taken from the same height H at a distance B apart.

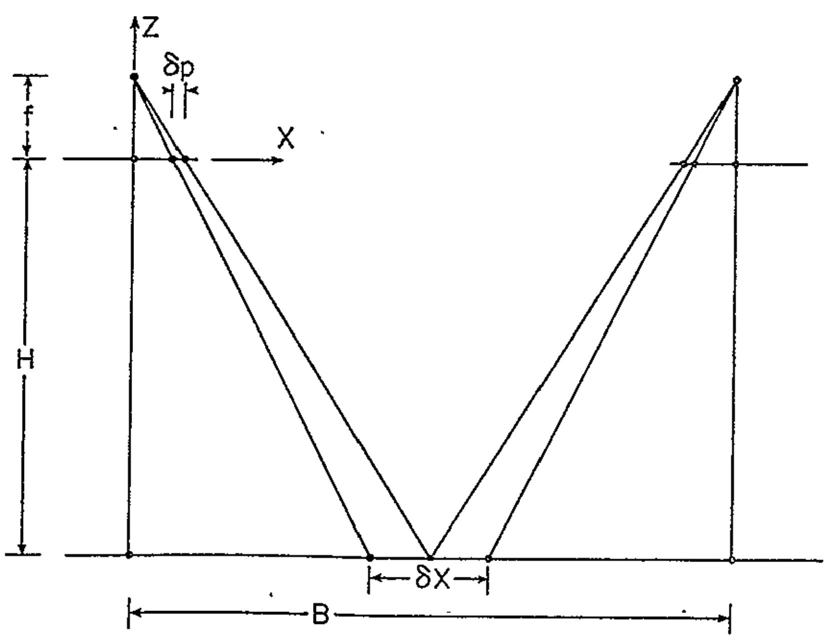
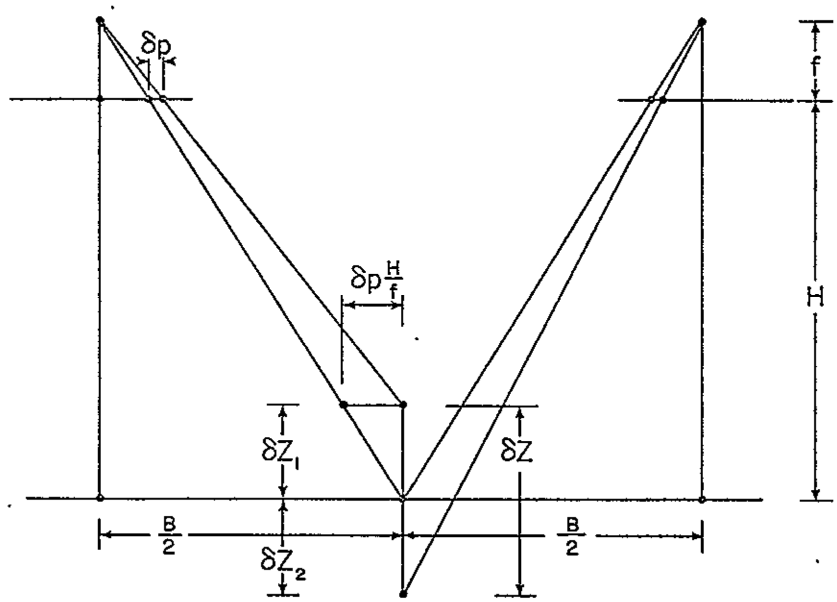
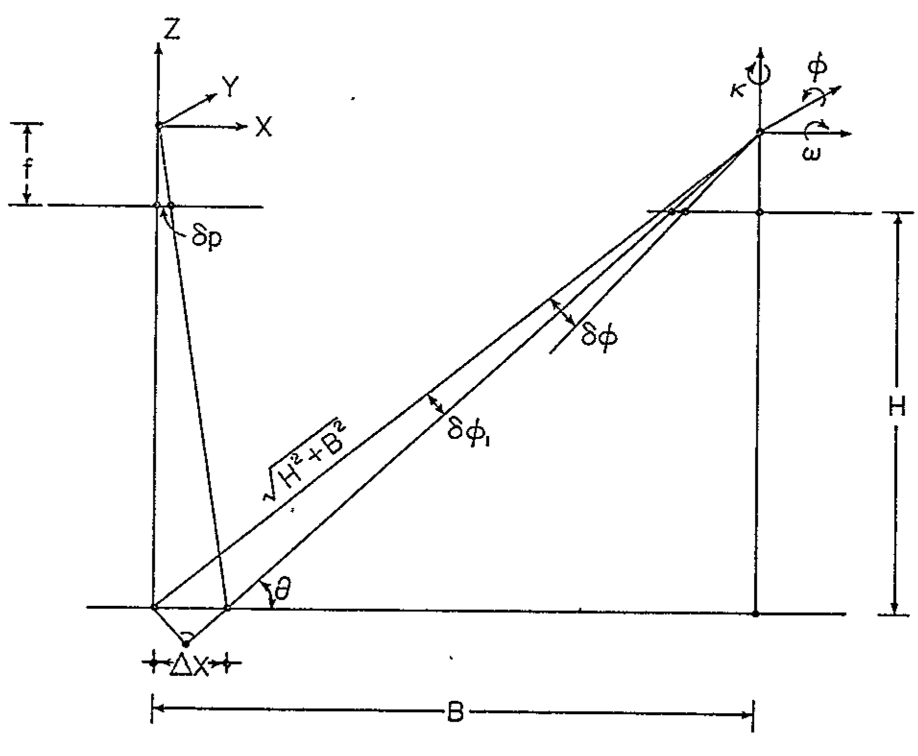


Fig. 2



δp
$\delta p \frac{H}{f}$
f
H
δZ_1
δZ
$\frac{B}{2}$
δZ_2
$\frac{B}{2}$

Fig. 3



Z
Y
X
f
δp
κ
φ
ω
δφ
H
$\delta\phi_1$
$\sqrt{H^2+B^2}$
θ
ΔX
B

Fig. 4

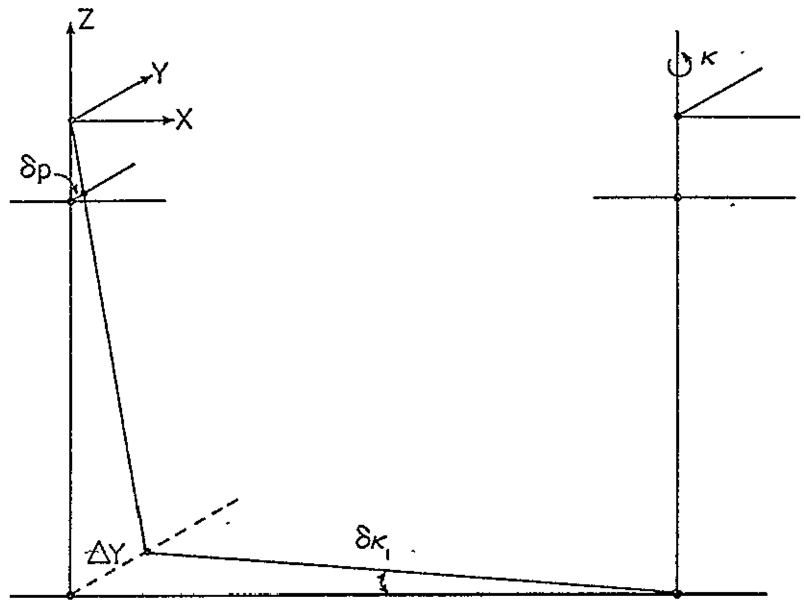


Fig. 5

therefore, the errors in the orientation angles φ and ω are

$$\delta\varphi = 2\delta\varphi_1 = \frac{2}{f}\,\frac{H^2}{H^2+B^2}\,\delta p = \delta\omega\ . \tag{4.4}$$

From Figure 5 the error in the orientation angle $\varkappa$ can be derived as follows:

$$B\,\delta\varkappa_1 = \Delta Y = \frac{H}{f}\,\delta p,$$

and thus

$$\delta\varkappa = 2\,\delta\varkappa_1 = \frac{2}{f}\,\frac{H}{B}\,\delta p\ . \tag{4.5}$$

In accordance with the definition in coordinate system (4), the errors in the coordinates Y and Z, contributed by $\delta\varkappa$ and $\delta\varphi$ will be

$$\delta Y_\varkappa = B \cdot \delta\varkappa \tag{4.6}$$

$$\delta Z_\varphi = B \cdot \delta\varphi \tag{4.7}$$

As a numerical example close to reality, let Figure 6 show a strip of photographs of semicircular shape over an arc of 180°.

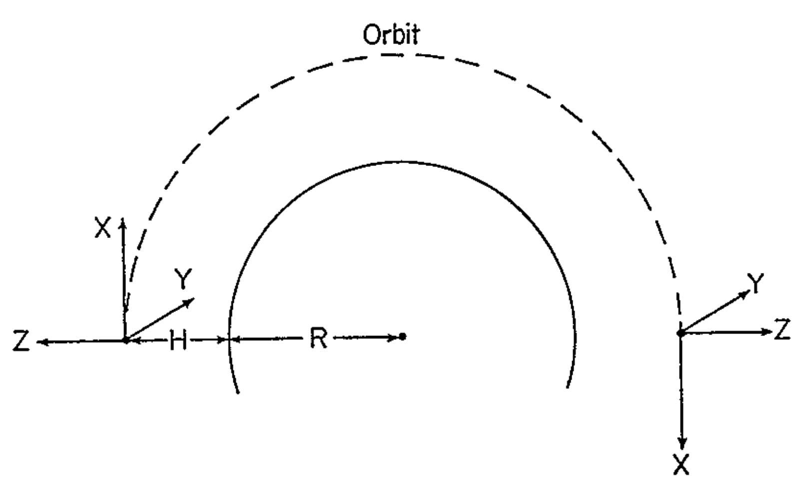


Fig. 6

The height is $H = 110$ km; the lunar radius is $R = 1738$ km and there is a total of 70 photographs in the strip so that $B \approx 83$ km. The focal length of the camera is 76 mm and the error of plate coordinate measurements is $\delta p = 5 \mu m$. The problem is to find the total accumulated errors in $\Delta X = \Sigma \delta X$, etc. at the end point of the strip.

For two successive photographs, equations 4.1 through 4.3 yield

$$\delta X = \delta Y = 14.47 \text{m}$$
$$\delta Z = 38.36 \text{m}$$

and equations 4.4 through 4.7 give

$$\delta \omega = \delta \varphi = 17\overset{''}{.}29$$
$$\delta \varkappa = 17\overset{''}{.}98$$

$$\delta Z_{\varphi} = 6.96 \text{ m}$$
$$\delta Y_{\varkappa} = 7.24 \text{ m}.$$

Thus for photography alone, the total accumulated errors at the end point (5805 km away) are:

$$\Delta X = \sum_{1}^{70} \delta X = 1.013 \text{ km} \qquad \text{or} \qquad 174 \text{ ppM}$$

$$\Delta Y = \sum_{1}^{70} (\delta Y + \delta Y_{\varkappa}) = 1.520 \text{ km} \qquad \text{or} \qquad 262 \text{ ppM}$$

$$\Delta Z = \sum_{1}^{70} (\delta Z + \delta Z_{\varphi}) = 3.170 \text{ km} \qquad \text{or} \qquad 546 \text{ ppM}.$$

The effect of altimetry is mainly in the Z direction. Since the altimetry error is $\delta a = 2$m, the error in Z for one pair of photographs becomes only

$$\delta z = 2 \, \delta a = 4 \text{ m},$$

and, therefore, the total error in Z reduces to

$$\Delta Z = \Sigma(\delta z + \delta Z\varphi) = 767.13\,\text{m} \text{ or } 132\,\text{ppM}$$

which is considerably smaller than the ΔZ from photography alone.

The stellar photography with an accuracy of 20″, does not permit the accumulation of errors in $\delta\kappa$, $\delta\varphi$, $\delta\omega$, thus $\delta Y_{\kappa} = \delta Z_{\varphi} = 0$, and therefore, the total errors in Y and Z, including the effects of photography and altimetry, will be

$$\Delta Y = \Sigma\delta Y = 1018.15\,\text{m} \text{ or } 174\,\text{ppM}$$

$$\Delta Z = \Sigma\delta Z = 280.0\,\text{m} \text{ or } 48\,\text{ppM}$$

again a considerable reduction. The total error in X will remain the same

$$\Delta X = 1013.15\,\text{m} \text{ or } 174\,\text{ppM}.$$

A displacement in X or Y, however, causes a displacement in orientation which, with respect to the selenocenter in the XZ plane is

$$\frac{1.013\,\text{km}}{1848.\,\text{km}} = 113.''06$$

and one of similar magnitude in the YZ plane. Such angular displacements are controlled by the stellar camera orientation and therefore the errors in X or Y are limited to $20'' \times (R + H)$, i.e.,

$$\Delta X = \Delta Y = 180\,\text{m} \text{ or } 31\,\text{ppM}.$$

These numbers together with the maximum expected $\Delta Z = 280\,\text{m}$ or 48ppM, represent reasonable estimates of the accumulated errors over the strip coming from a $5\,\mu\text{m}$ photo-measure error in each photograph.

5. REDUCTION AND MODELING OF THE OBSERVATIONS AND THE USE OF ORBITAL INFORMATION

5.1 The Use of Observations from the Mapping Camera System

Since the terrain camera is separately calibrated, the coordinates used in the analysis are not the measured photo-images coordinates, but those which have been corrected for distortion, film shrinkage, etc., i.e., they are assumed to be free of systematic errors and as such they fit the simple photogrammetric model of the following projection equations:

$$
\begin{aligned}
f_{x_{ij}} &= x_{ij} - x_0 + f\,\frac{P_{ij}}{S_{ij}} = 0 \\
f_{y_{ij}} &= y_{ij} - y_0 + f\,\frac{Q_{ij}}{S_{ij}} = 0
\end{aligned}
\tag{5.1}
$$

In the above equations

$$
\begin{bmatrix} P_{ij} \\ Q_{ij} \\ S_{ij} \end{bmatrix} = M_j \begin{bmatrix} \Delta X_{ij} \\ \Delta Y_{ij} \\ \Delta Z_{ij} \end{bmatrix}
$$

where $M_j = R_3(\varkappa_j)\,R_2(\varphi_j)\,R_1(\omega_j)$

$$
\begin{bmatrix} \Delta X_{ij} \\ \Delta Y_{ij} \\ \Delta Z_{ij} \end{bmatrix} = \begin{bmatrix} X_i - X_j \\ Y_i - Y_j \\ Z_i - Z_j \end{bmatrix}
$$

X_i, Y_i, Z_i are the coordinates of i^{th} ground point,

X_j, Y_j, Z_j, ω_j, φ_j, $\varkappa_j$ are the coordinates and orientation angles of j^{th} camera exposure station,

x_{ij} , y_{ij} are the plate coordinates and

x_0 , y_0 , f are the plate coordinates of the principal point and the focal length, all determined from calibration.

The altimeter observations are modeled in a straightforward way, as follows:

$$f_{d_j} = d_{ij} - [(X_i - X_j)^2 + (Y_i - Y_j)^2 + (Z_i - Z_j)^2]^{\frac{1}{2}} = 0 \qquad (5.2)$$

where d_{ij} is the measured distance between the j^{th} camera exposure station and the corresponding (illuminated) i^{th} point on the ground.

The stellar camera produces observations with respect to a star catalog system, which subsequently will be reduced to a moon-fixed system as shown below. Using a libration theory we have

$\overline{X}_{cj}$	=	vector in the camera axes system at the j^{th} exposure
$\overline{X}_I$	=	same vector in the inertial system
$\overline{X}_S$	=	same vector in selenocentric moon-fixed system used in the libration theory
K_j , Φ_j , Ω_j	=	camera orientation angles with respect to the inertial system at the j^{th} exposure
$\varkappa_j$, φ_j , ω_j	=	same angles with respect to the selenocentric moon-fixed system.

We have

$$\overline{X}_{cj} = \widetilde{M}(K_j , \Phi_j , \Omega_j)\overline{X}_I$$

$$\overline{X}_{cj} = M(\varkappa_j , \varphi_j , \omega_j)\overline{X}_S .$$

At epoch t_j the three Eulerian angles from the libration theory are

$$\epsilon_j = \epsilon(t_i) , \quad \theta_j = \theta(t_j) , \quad \psi_j = \psi(t_j)$$

and $$\overline{X}_S = R_3(\psi_j)R_1(-\theta_j)R_3(\epsilon_j)\overline{X}_I$$

or,

$$\overline{X}_S = R(\psi_j , \theta_j , \epsilon_j)\overline{X}_I$$

and $$\overline{X}_{cj} = M(\varkappa_j , \varphi_j , \omega_j)R(\psi_j , \theta_j , \epsilon_j)\overline{X}_I .$$

Also $\widetilde{M}(K_j, \Phi_j, \Omega_j) = M(\varkappa_j, \varphi_j, \omega_j) R(\psi_1, \theta_1, \epsilon_j)$

and $M(\varkappa_j, \varphi_j, \omega_j) = \widetilde{M}(K_j, \Phi_j, \Omega_j) R_3(-\epsilon_j) R_1(\theta_j) R_3(-\psi_j) =$

$$= \widetilde{M}(K_j, \Phi_j, \Omega_j) \widetilde{R}(\psi_j, \theta_j, \epsilon_j).$$

After the matrix M has been computed, the angles $\varkappa_j$, φ_j, ω_j can be calculated from the elements of M as follows:

$$\varkappa_j = \tan^{-1}\left(\frac{-m_{21}}{m_{33}}\right)$$

$$\omega_j = \tan^{-1}\left(\frac{-m_{31}}{m_{33}}\right)$$

$$\varphi_j = \tan^{-1}\left(\frac{m_{31}}{m_{32}^2 + m_{33}^2}\right)$$

The variance-covariance matrix of the quantities $\varkappa_j$, φ_j, ω_j, as computed from the variances and covariances of K_j, Φ_j, Ω_j, ψ_j, θ_j, ϵ_j given in Appendix B.

5.2 General Comments on the Use of Orbital Information

The analysis of tracking data for orbit determination depends heavily on the gravity field model used. The reference system naturally suited for such an orbit analysis is a selenocentric inertial system. The analysis of earth-based tracking data involves the rotation of the earth, earth station coordinates and the lunar theory. The role of libration theory is limited to the description of variations in the moon's gravity field due to the moon's rotation with respect to the inertial selenocentric system.

The geometry of the observations (range, range rate) is poor since tracking takes place always from the same direction and it is interrupted while the lunar satellite is on the far side of the moon. This interruption of tracking and the fact that the knowledge of gravity information on the far side is still rather poor,

impose strong limitations on the use of long arcs of more than one revolution.

For the purpose of this analysis, side overlaps of photo-strips from different passes provide better quality ties between the network points than one could hope to get through long arc techniques. Each orbital arc corresponding to one photo-strip is viewed as a separate orbit and tracking provides information on the geometry of camera points along each arc.

Both range and range rate observations can provide relatively good estimates on the shape and scale of each arc, but not on rotational displacements with the earth (geocenter or tracking station) as a center. Positioning with respect to the selenocenter and orientation mainly come from the lunar gravity model. The weakness of the orbit in positioning and especially in orientation may well not be reflected in the statistics (variances and covariances) of orbit point coordinates. The reason is that statistics strongly depend on the model, and they are realistic only as far as the model is realistic. While the shape and scale of the orbit mostly depend on the observations themselves for which realistic a priori statistics are available; the truncation of the gravity field and its uncertainties make the statistics of gravity field dependent parameters (position and orientation) unrealistic. Past phototriangulation solutions with use of orbital support indeed show displacements in the "adjusted" orbit, mainly in rotations of the arcs around some point in the orbit (see e.g., Figures 5 and 6 in [7]).

The conclusion is that in the use of orbital support in phototriangulation, one needs to consider two deviations from the classical implementation of a priori estimates and their statistics. First, different passes will have to be freed from their reference to a common moon-fixed system. Second, since statistics of orbit positions are not reliable, they need to be updated in a proper way. In simple terms, the orbit information has to be properly weighted relative to the photographic information. In the next chapters these two problems are investigated.

6. STATISTICAL COMPARISON OF DATA

6.1 The Input Data, Their Contributions and Their Associated Reference Systems

Table 1 is a summary of the various data types and where they contribute to the problem of determination of shape, scale, orientation and positioning of a selenodetic network.

Table 1

Type of Data	Shape	Scale	Orientation	Positioning
Lunar Photography	YES			
Stellar Photography	YES		YES	
Altimetry	YES	YES		
Orbit State Vectors	YES	YES	YES	YES

Since all data types contribute to the shape of the network, it becomes a possible tool for detecting inconsistencies between the various types of data. Positioning depends only on orbital support and one can expect the tie between the selenodetic network and the selenocenter to be only as strong as provided by the orbit analysis. Orientation depends both on stellar photography and on the orbit, but the low quality of stellar photography makes the detection of possible inconsistencies in orbit orientation with respect to a moon-fixed system almost impossible.

In this work the target of the analysis is only the determination of the shape and scale of the network, and for positioning and orientation, one has to rely on orbital support. The latter also provides a means for scale information, and as such it can detect scale inconsistencies in the altimetry data.

To avoid inconsistencies in the orientation and positioning between the different passes and also between these passes and the corresponding stellar photography, in solutions directed to determine the shape and scale of the selenodetic network, the stellar photography and the state vectors have to be freed from their dependence on their given system of reference. This is possible with the use of minimal

constraints particular to each type of data. For example, instead of absolute camera orientation, one can use relative camera orientation, i.e., the orientation of the camera axes at each exposure with respect to that at the time of the first exposure in each strip. These relative orientations are coordinate system independent. For the use of orbital support, the approach is to first establish an arbitrary system for the network through minimal constraints. Then positional and orientation parameters with respect to this system are introduced for each pass as additional parameters. From these solved parameters one may detect either the random or systematic pattern in the orientation and position of the orbit systems. If the differences between these position and orientation parameters are statistically significant, then the approach of separating the coordinate systems for each arc is justified.

6.2 Method for the Statistical Comparison of Data

6.2.1 Statistical Test for the Recovery of Inconsistencies Between Two Data Groups

It has already been shown that when combining lunar terrain photography with any of the other observational groups (stellar photography, altimetry and orbital support), there exists a set of parameters that can be determined (estimated). At least the shape can always be determined, in which case the estimable parameters are the coordinates with respect to a system established by minimal constraints in which the scale is also constrained.

To generalize, consider two groups of observations G_1 and G_2 and a set of estimable parameters X, common to both groups. If X_1 and X_2 are unbiased linear estimates of X, using groups G_1 and G_2, respectively, and if the groups are divided into

$$G_1 = \left\{L_0, L_1\right\} ,$$

$$G_2 = \left\{L_0, L_2\right\} ,$$

where L_0 are observations common to both groups, then

$$X_1 = Q_{01} L_0 + Q_1 L_1 ,$$
$$X_2 = Q_{02} L_0 + Q_2 L_2 , \tag{6.1}$$

where Q_{01} , Q_{02} , Q_1 , Q_2 are known matrices (available from the estimation) of appropriate dimensions.

Next consider the matrix

$$X_D = X_1 - X_2 = [I , -I] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

The observations L_0 , L_1 , L_2 are taken to be independent with their respective variance-covariance matrices Σ_0 , Σ_1 , Σ_2 .

Then

$$X_D = [I \mid -I] \begin{bmatrix} Q_{01} & Q_1 & 0 \\ Q_{02} & 0 & Q_2 \end{bmatrix} \begin{bmatrix} L_0 \\ L_1 \\ L_2 \end{bmatrix}$$

$$= [(Q_{01} + Q_{02}) \mid Q_1 \mid - Q_2] \begin{bmatrix} L_0 \\ L_1 \\ L_2 \end{bmatrix} = QL . \tag{6.2}$$

The variance-covariance matrix of X_D is

$$\Sigma_D = Q \Sigma_L Q^T$$

$$= [(Q_{01} + Q_{02}) \quad Q_1 \quad - Q_2] \begin{bmatrix} \Sigma_0 & 0 & 0 \\ 0 & \Sigma_1 & 0 \\ 0 & 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} (Q_{01} + Q_{02})^T \\ Q_1^T \\ -Q_2^T \end{bmatrix} =$$

$$= (Q_{01} - Q_{02}) \Sigma_0 (Q_{01} - Q_{02})^T + Q_1 \Sigma_1 Q_1^T + Q_2 \Sigma_2 Q_2^T \tag{6.3}$$

Further assume that L_0, L_1 , L_2 are normally distributed. This is a reasonable assumption since L_0 , L_1 , L_2 are either observations corrupted by Gaussian noise or their linear transformations (synthetic observations). To indicate this, one can write

$$L_1 \sim N (\mu_1 , \Sigma_1) , \; L_2 \sim N (\mu_2 , \Sigma_2) , \; L_0 \sim N (\mu_0 , \Sigma_0)$$

where $$\mu_0 = E(L_0), \ \mu_1 = E(L_1), \ \mu_2 = E(L_2)$$

$$\Sigma_0 = E\left\{(L_0 - \mu_0)(L_0 - \mu_0)^T\right\}$$

$$\Sigma_1 = E\left\{(L_1 - \mu_1)(L_1 - \mu_1)^T\right\}$$

$$\Sigma_2 = E\left\{(L_2 - \mu_2)(L_2 - \mu_3)^T\right\}$$

$E\{\ \}$ is the expectation operator and Σ_0, Σ_1, Σ_2 are assumed to be the true variance-covariance matrices of L_0, L_1, L_2.

Since X_D is a linear transformation of L_0, L_1, L_2, it is also normally distributed, thus

$$X_D \sim N(\mu_D, \Sigma_D) \ .$$

Since X_1, X_2 are unbiased estimates of the same parameter vector,

$$E\{X_1\} = E\{X_2\} \ ,$$

and therefore,

$$\mu_D = E\{X_D\} = E\{X_1 - X_2\} = E\{X_1\} - E\{X_2\} = \overline{0}$$

and

$$X_D \sim N(\overline{0}, \Sigma_D) \ .$$

At this point a statistical test is performed on the hypothesis:

H:X_D is a sample drawn from a population with the distribution $N(\overline{0}, \Sigma_D)$.

The statistical test is performed at a certain significance level, i.e., the hypothesis, that X_D belongs to the population with distribution $N(0, \Sigma_D)$ with a probablility α, where $0 \leq \alpha \leq 1$, is tested. The hypothesis is certainly rejected for $\alpha = 1$ and certainly accepted for $\alpha = 0$. The choice of α, from the interval $0 < \alpha < 1$, is subjective, but it has to be close to 1 to be meaningful. Standard choices are $\alpha = 95/100$ and $\alpha = 99/100$. For the performance of the test see [1], especially chapter 5.

Since X_D is intended to be an estimate of a null vector of quantites, it is known a priori that the above hypothesis H is ture, and its rejection in the performance of the statistical test would correspond to a rejection of the estimate X_D

or its variance-covariance matrix Σ_0. In the latter case one would have to accept the fact that the observations L_0, L_1, L_2 and their statistics Σ_0, Σ_1, Σ_2 are statistically inconsistent at the significance level of the test. The reason for this inconsistency can be either a systematic error in at least one of the observation sets L_0, L_1, L_2 or one or more erroneous variance-covariance matrices from Σ_0, Σ_1, Σ_2. Systematic error means that an actual observation does not correspond to its expected value included in the model.

6.2.2 Testing for Inconsistencies due to Systematic Errors in Altimetry and Stellar Photography

For the particular problem in this paper, consider the following set of observations:

$$G_L = \{\text{lunar photographic observations}\}$$

$$G_S = \{\text{stellar camera observations}\}$$

$$G_A = \{\text{altimeter observations}\}$$

$$G_0 = \{\text{a priori estimates of camera exposure station coordinates from orbit analysis}\}$$

Assume that group G_L is free of systematic errors and that the corresponding variance-covariance matrix does not deviate significantly from its true value. This assumption is reasonable in view of previous photogrammetric experience.

Next consider the possibility of systematic errors in groups G_S and G_A. Testing the groups, $G_1 = G_L$ and $G_2 = G_L \cup G_S$, one can detect systematic errors in G_S. The common parameters used in the test will be the network coordinates with respect to an arbitrary system established through minimal constraints specifying position and scale.

In a similar way one can test the groups $G_1 = G_L$ and $G_2 = G_L \cup G_A$. However, from these tests one cannot detect significant systematic errors, such as a scale factor in altimeter observations or a constant error in orientation of the stellar camera. To detect such errors one needs comparison with

orbital support, and for this purpose one should test groups like $G_1 = G_L \cup G_S$, $G_2 = G_L \cup G_0$ for orientation; or $G_1 = G_L \cup G_A$, $G_2 = G_L \cup G_0$ for scale. Although such tests can detect orientation or scale errors, they cannot distinguish between stellar camera errors and altimeter observations, or errors in the orbital support. This is a definite limitation of the system.

6.2.3 Testing for Inconsistencies in System Orientation and Positioning among Different Orbit Passes

As mentioned earlier in view of the geometry of earth-based tracking used in orbital analysis, it is expected that shape and scale for each arc is well determined, but there is a possibility of weak determination in positioning and especially in orientation. To investigate this matter one can test the groups $G_1 = G_L \cup G_0^i \cup G_0^j$ and $G_2 = G_L \cup \tilde{G}_0^i \cup \tilde{G}_0^j$, where G_0^i and G_0^j are given orbit coordinates for the i^{th} and j^{th} arcs; while $\tilde{G}_0^i$ and $\tilde{G}_0^j$ are coordinates of the arcs where the possibility of a change of coordinate system has been introduced in the model. If no inconsistencies are justified (i.e., our hypothesis is not rejected), one can proceed in testing the possibility of different scales in altimetry and orbital support. The appropriate test for this is between the groups

$$G_1 = G_L \cup (\bigcup_{i=1}^{n} G_0^i) \quad \text{and} \quad \overline{G_2} = G_L \cup (\bigcup_{i=1}^{n} G_0^i) \cup G_A .$$

In case the arcs don't fit together one would rather test

$$G_1 = G_L \cup (\bigcup_{i=1}^{n} \tilde{G}_0^i) \quad \text{and} \quad G_2 = G_L \cup (\bigcup_{i=1}^{n} \tilde{G}_0^i) \cup G_A .$$

Finally, if there is no orientation inconsistency between different orbit arcs, it is possible to test for stellar camera orientation only, in which case one tests the groups

$$G_1 = G_L \cup (\bigcup_{i=1}^{n} G_0^i) \quad \text{and} \quad G_2 = G_L \cup (\bigcup_{i=1}^{n} G_0^i) \cup G_S .$$

At this point it should be obvious that in the search for statistical inconsistencies between the various types of data, one is faced with an enormous, although not impossible, computational task. The problem obviously calls for a simplified, not

completely rigorous approach. The main problem is to determine whether or not the various orbit arcs are consistent in position and orientation. After this question has been answered, one has to find a way to take care of the inconsistencies between the two groups, namely the terrain photography with its supporting observations ($G_L \cup G_S \cup G_A$) and the estimates of orbit coordinates (G_O) on the various passes which may be constrained to the same reference system or not.

Before continuing with this problem one should note that if inconsistencies in system definition between the different passes are found, one must abandon the hope of positioning or orienting (except via stellar camera observations) the network in a meaningful way. The simplest way to check reference system consistency between orbit arcs is given below.

First of all, a solution must be made using terrain photography and all passes to determine coordinates with respect to the system specified by one of the arcs. All other arcs are considered to refer to a different system which differs from the first one by a vector δ_i, where $\delta_i^T = [\delta_{iS}^T \mid \delta_{iP}^T]$ corresponds to the i^{th} arc ($i = 2, 3, \cdots n$) and δ_{iS}, δ_{iP} are vectors of shifts and orientation angles included as parameters and solved for.

Next, a series of tests are performed on the hypotheses as follows:

$$H_{iS}: \delta_{iS} \text{ belongs to the population } N(0, \Sigma_{\delta_{iS}})$$

$$H_{iR}: \delta_{iR} \text{ belongs to the population } N(0, \Sigma_{\delta_{iR}})$$

$$\text{for } i = 2, 3, \cdots n$$

where $\Sigma_{\delta_{iS}}$, $\Sigma_{\delta_{iR}}$ are the variance-covariance matrices of δ_{iS} and δ_{iR}, also computed during the solution.

If all hypotheses are accepted then one can obviously constrain all the arcs to the same system. If the majority of the tests is accepted than the coordinate system of these arcs is used as the reference system and all the remaining arcs are free of the coordinate system.

6.2.4 Testing for Inconsistencies between Photogrammetric Control and Orbital Support Data

After the problem of the different passes has been settled, one can return to the problem of taking care of inconsistencies between photogrammetric control and orbital support. Returning to the test of the hypothesis $X_D \sim N(\overline{0}, \Sigma_D)$, note that for every significant level $\alpha < 1$, one can find a scalar σ such that the hypothesis $X_D \sim N(0, \Sigma_D')$, where $\Sigma_D' = \sigma \Sigma_D$, is accepted.

It is obvious that an increase in Σ_D' makes the vector X_D a more possible candidate as a member of the population $N(0, \Sigma_D')$.

If the test refers to two groups of observations of the form $G_1 = L_1$ and $G_2 = \left\{L_1, L_2\right\}$, then

$$X_1 = Q_{11} L_1$$

$$X_2 = Q_{21} L_1 + Q_2 L_2$$

and

$$X_D = X_1 - X_2 = (Q_{11} - Q_{21}) L_1 + Q_2 L_2 = [\,(Q_{11} - Q_{21}) \mid Q_2\,] \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$$

Setting $(Q_{11} - Q_{21}) = Q_1$, one gets

$$X_D = Q_1 L_1 + Q_2 L_2$$

and

$$\Sigma_D = Q_1 \Sigma_1 Q_1^T + Q_2 \Sigma_2 Q$$

One can always find two constants σ_1, σ_2 such that the hypothesis $X_D \sim N(0, \Sigma_D')$ is accepted where

$$\Sigma_D' = Q_1 (\sigma_1 \Sigma_1) Q_1^T + Q_2 (\sigma_2 \Sigma) Q_2^T$$

This indicates that the effect of the inconsistency in the data can be reduced by updating the variance-covariance matrices. Obviously this cannot be done in an arbitrary way and a proper way has to be found, conforming with some criterion of optimality.

In simple terms, one needs to find a proper way of weighting orbital support versus lunar photography for a statistically consistent solution. This problem is investigated in detail in the next chapter.

6.2.5 The Use of Minimal and Inner Constraints

The previous discussion repeatedly mentioned the use of a set of minimal constraints as a means to find station coordinates even though the observations may be position, orientation or even scale invariant.

Minimal constraints are a very useful tool, but there are some computational problems associated with their use. To understand this, consider an analytic phototriangulation for a strip of photographs where the camera coordinates and orientation angles of the first camera exposure point in the strip are constrained to a set of constants, and scale is provided by some other means, e.g., altimeter observations. It is obvious that in the solution with coordinates as unknowns, the variances of the first camera point will be zero. Furthermore, the variances of point coordinates will grow larger for points further and further away from the first point. Since the variance-covariance matrix of the unknowns is the inverse of the coefficient matrix in the normal equations, any difficulties in inverting such a matrix will be present in the inversion of its inverse, i.e., of the normal equations' coefficient matrix. A matrix that has diagonal elements which grow systematically along the diagonal, is bound to be difficult to invert, compared to a matrix that has a smaller possible variation along the diagonal.

Since a different set of minimal constraints will result in a different coefficient matrix in the normal equations, it is obvious that one should use a set of minimal constraints which is optimal from a computational point of view. When the optimality criterion is the minimum trace of the variance-covariance matrix (inverse of the normal equations' coefficient matrix), the corresponding set of optimum minimal constraints is the so-called "inner constraints." This concept of inner constraints has been investigated in [2], with emphasis on range observations.

Appendix A is an extension of this concept for photogrammetric application. The use of an inner set of constraints instead of a more easily identified set of minimal constraints causes no additional difficulties. Formulas for the transformation of solution vectors and variance-covariance matrices when different sets of minimal constraints are used, are found in [14].

7. OPTIMAL WEIGHTING OF ORBITAL SUPPORT DATA VERSUS LUNAR TERRAIN PHOTOGRAPHY

In this chapter a way is sought to use two sets of observations together with their variance-covariance matrices, when the two sets are statistically inconsistent (at a certain significance level). The only rigorous answer, of course, is to reject either or both sets of observations, the former if there is reason to believe that the inconsistency comes from that set only. Such a negative attitude does not solve the problem, and instead one should look for a way of using the observations despite their inconsistency.

If there is no way of improving one or both sets of observations, then the effect of the inconsistency can be reduced by updating their variance-covariance matrices. The simplest way of updating thses matrices would be to assume that they differ from their consistent counterparts by scalar multiplications only. This way the relative accuracies between the observations within each set would be preserved, while the two sets are properly weighted against each other by giving the less consistent set a smaller weight, thus reducing its effect in the solution.

More explicitly, given two sets of observations L_1 and L_2, with the corresponding variance-covariance matrices K_1 and K_2, it is assumed that their true variance-covariance matrices are $\Sigma_1 = \sigma_1 K_1$ and $\Sigma_2 = \sigma_2 K_2$, where the scalars σ_1 and σ_2 are to be determined. Since there are an infinite number of such scalars which would reduce the effect of inconsistency in the two sets of observations, some criterion is needed to determine the optimum pair of values for σ_1 and σ_2. In the following, two such criteria are discussed, namely the well known Maximum Likelihood and the less familiar but in this case computationally more practical Minimum Norm (MINQUE).

7.1 Maximum Likelihood Criterion

Lemma 1 [1]

If $x = (x_1, x_2, \cdots x_n)^T$ is a random vector variable with expectation $E(x) = \mu$, and its variance-covariance matrix is

$$E\left\{(x - \mu)(x - \mu)^T\right\} = V \ ,$$

then assuming that x is multivariate normally distributed, the joint probability density function is given by

$$f(x_1, x_2, \cdots x_n) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{|V|}} \exp\left[- \frac{(x - \mu)^T V^{-1}(x - \mu)}{2}\right] \quad (7.1)$$

□

7.1.1 The Regression Model

Suppose that a set of parameters X_a and two sets of observations L_{a1} and L_{a2} are related by the vector model $\bar{f}(X_a, L_{a1}, L_{a2}) = 0$. After a Taylor series expansion and neglection of second or higher order, the following set of linear equations is obtained:

$$0 = \bar{f}(X_0, L_{b1}, L_{b2}) + \left.\frac{\partial \bar{f}}{\partial X_a}\right|_{\substack{0\\b}} (X_a - X_0) + \left.\frac{\partial \bar{f}}{\partial L_{a1}}\right|_{\substack{0\\b}} (L_{a1} - L_{b1}) +$$

$$+ \left.\frac{\partial \bar{f}}{\partial L_{a2}}\right|_{\substack{0\\b}} (L_{a2} - L_{b2}) \ ,$$

where X_0 are approximate values of X_a; L_{b1}, L_{b2} are observed values of L_{a1}, L_{a2} and the partials are evaluated for approximate and observed values. The above equation may be written as

$$0 = -W + AX + B_1 V_1 + B_2 V_2$$

or

$$W = AX + B_1 V_1 + B_2 V_2 \ . \quad (7.2)$$

The following information is also known:

$$E(V_1) = 0 \qquad\qquad E(V_2) = 0$$

$$\mathrm{Var}(V_1) = E\left\{V_1 V_1^T\right\} = \sigma_1^2 K_1$$

$$\mathrm{Var}(V_2) = E\left\{V_2 V_2^T\right\} = \sigma_2^2 K_2$$

where Var (y) stands for the variance-covariance matrix of vector y; $E\{\ \}$ is the expectation operator; K_1 , K_2 are known positive-definite matrices and σ_1^2 , σ_2^2 are the unknown scalars to be determined.

Some of the obvious results are

$$E(W) = E(AX) + E(BV_1) + E(BV_2) = AX \, .$$

Also:

$$\mathrm{Var}(W) = E\left\{(W - AX)(W - AX)^T\right\} =$$

$$= E\left\{(B_1 V_1 + B_2 V_2)(B_1 V_1 + B_2 V_2)^T\right\} =$$

$$= B_1 E\left\{V_1 V_1^T\right\} B_1^T + B_1 E\left\{V_1 V_2^T\right\} B_2^T +$$

$$+ B_2 E\left\{V_2 V_1^T\right\} B_1^T + B_2 E\left\{V_2 V_2^T\right\} B_2^T \, .$$

If V_1 and V_2 are uncorrelated, we have

$$\mathrm{Var}(W) = B_1 \mathrm{Var}(V_1) B_1^T + B_2 \mathrm{Var}(V_2) B_2^T =$$

$$= \sigma_1^2 B_1 K_1 B_1^T + \sigma_2^2 B_2 K_2 B_2^T =$$

$$= \sigma_1^2 S_1 + \sigma_2^2 S_2 =$$

$$= \sigma_1^2 (S_1 + \gamma_2 S_2) = \sigma_1^2 H$$

where $\gamma_2 = \left(\frac{\sigma_2}{\sigma_1}\right)^2$.

If σ_1^2 and σ_2^2 were known, then the least squares estimator $\hat{X}$, of X, is given from the solution of the normal equations

$$[A^T(\sigma_1^2 S_1 + \sigma_2^2 S_2)^{-1} A]\,\hat{X} = A^T(\sigma_1^2 S_1 + \sigma_2^2 S_2)^{-1}\, W$$

and $\hat{X}$ minimizes the quadratic form

$$\varphi = \frac{1}{\sigma_1^2} V_1^T K_1^{-1} V_1 + \frac{1}{\sigma_2^2} V_2^T K_2^{-1} V_2$$

7.1.2 The Maximum Likelihood Solution

The problem is now to find from all possible values of σ_1^2 and σ_2^2, the optimum pair, where the optimization criterion remains to be established. Recalling the definition of the vector W,

$$W = -\bar{f}\,(X_0, L_b) \qquad L_b^T = [\,L_{b1}^T \mid L_{b2}^T\,] \;,$$

and noticing that W is a deterministic function of the random observations L_b, the established criterion of optimization is the "maximum likelihood" criterion, where the likelihood is referred to the vector W.

The likelihood of W is given by $L = f(w_1, w_2, \cdots w_n)$, where f is the joint probability density function of the components w_1, $w_2, \cdots w_n$ of the vector W. Equation (7.1), after replacing x with W, μ with AX and V with $\sigma_1^2 H$, yields

$$L = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{|\sigma_1^2 H|}} \exp\left[\frac{(W - AX)^T(\sigma_1^2 H)^{-1}(W - AX)}{2}\right]$$

To get a maximum of L we can alternatively get a maximum for

$$\lambda = \log L \;,$$

and

$$\lambda = -\log(2\pi)^{\frac{n}{2}} - \log\sigma_1^n - \log\sqrt{|H|} - \frac{(W - AX)^T H^{-1}(W - AX)}{2\sigma_1^2}$$

where $\sqrt{|\sigma_1^2 H|}$ was replaced by $\sigma_1^n\sqrt{|H|}$.

The function λ obtains a maximum for these values of X, σ_1, γ_2 which satisfy the following equations

$$\frac{\partial\lambda}{\partial X} = 0 \;, \quad \frac{\partial\lambda}{\partial\sigma} = 0 \;, \quad \frac{\partial\lambda}{\partial\gamma} = 0 \;.$$

The developed form of these partial derivatives are as follows:

$$\frac{\partial \lambda}{\partial X} = -\frac{1}{2\sigma_1^2}\frac{\partial}{\partial X}[(W - AX)^T H^{-1}(W - AX)] =$$

$$= -\frac{1}{2\sigma_1^2}\frac{\partial}{\partial X}[W^T H^{-1} W - 2W^T H^{-1} AX + X^T A^T H^{-1} AX] =$$

$$= -\frac{1}{2\sigma_1^2}[-2W^T H^{-1} A + 2X^T A^T H^{-1} A] = 0,$$

where the following identities have been used:

$$\frac{\partial}{\partial y}(Ry) = R \quad \text{and} \quad \frac{\partial}{\partial y}(y^T R y) = 2y^T R$$

For $\sigma_1^2 \neq 0$ and $\sigma_1^2 < \infty$, $\frac{\partial \lambda}{\partial X} = 0$ gives:

$$\hat{X} = (A^T H^{-1} A)^{-1} A^T H^{-1} W \tag{7.3}$$

The derivative with respect to σ_1 is

$$\frac{\partial \lambda}{\partial \sigma_1} = -\sigma_1^{-n}\frac{\partial(\sigma_1^n)}{\partial \sigma_1} - \frac{1}{2}\frac{\partial(\sigma_1^{-2})}{\partial \sigma_1}[(W - AX)^T H^{-1}(W - AH)] =$$

$$= -\frac{n}{\sigma_1} + \frac{1}{\sigma_1^3}(W - AX)^T H^{-1}(W - AX) = 0 .$$

From where for $\sigma_1 \neq 0$, one gets

$$\sigma_1^2 = \frac{(W - AX)^T H^{-1}(W - AX)}{n} \tag{7.4}$$

At this point, before proceeding further, two algebraic lemmata are introduced.

<u>Lemma 2</u>

For the square matrix $H = H(\gamma)$, the following relationship holds:

$$\frac{\partial}{\partial \gamma}[\log |H|] = \operatorname{tr}(H^{-1}\frac{\partial H^T}{\partial \gamma}) \tag{7.5}$$

Proof:

Based on [5], page 266, eq. (10.8.22), for a square matrix H

$$\frac{\partial |H|}{\partial \gamma} = \operatorname{tr}\left[H^* \frac{\partial H^T}{\partial \gamma} \right]$$

where H^* is the adjugate matrix of H (i.e., H^*_{ij} = cofactor of H_{ij}). From [19], page 39

$$H H^* = |H| I$$

where I is the identity matrix, or $H^* = |H| H^{-1}$. Therefore,

$$\frac{\partial}{\partial \gamma}[\log |H|] = \frac{1}{|H|} \frac{\partial |H|}{\partial \gamma} = \frac{1}{|H|} \operatorname{tr}\left[H^* \frac{\partial H^T}{\partial \gamma} \right] =$$

$$= \frac{1}{|H|} \operatorname{tr}\left[|H| H^{-1} \frac{\partial H^T}{\partial \gamma} \right] = \operatorname{tr}\left[H^{-1} \frac{\partial H^T}{\partial \gamma} \right].$$

□

Lemma 3

For the square non-singular matrix $H = H(\gamma)$, the following relationship holds:

$$\frac{\partial (H^{-1})}{\partial \gamma} = - H^{-1} \frac{\partial H}{\partial \gamma} H^{-1} . \tag{7.6}$$

Proof: $H \cdot H^{-1} = I$, $\frac{\partial I}{\partial \gamma} = \overline{0} = \frac{\partial H}{\partial \gamma} H^{-1} + H \frac{\partial (H^{-1})}{\partial \gamma}$

$$H \frac{\partial (H^{-1})}{\partial \gamma} = - \frac{\partial H}{\partial \gamma} H^{-1} \quad \text{and} \quad \frac{\partial (H^{-1})}{\partial \gamma} = -H^{-1} \frac{\partial H}{\partial \gamma} H^{-1} .$$

□

The partial derivative of λ with respect to γ_2 is

$$\frac{\partial \lambda}{\partial \gamma_2} = - \frac{1}{2} \frac{\partial}{\partial \gamma_2}[\log |H|] - \frac{1}{2\sigma_1^2} (W - AX)^T \frac{\partial (H^{-1})}{\partial \gamma_2} (W - AX) =$$

$$= - \frac{1}{2} \operatorname{trace}\left(H^{-1} \frac{\partial H}{\partial \gamma_2}\right) + \frac{1}{2\sigma_1^2} (W - AX)^T H^{-1} \frac{\partial H}{\partial \gamma_2} H^{-1} (W - AX) .$$

Since $$H = S_1 + \gamma_2 S_2 ,$$

$$\frac{\partial H}{\partial \gamma_2} = S_2 \quad \text{and}$$

$$\frac{\partial \lambda}{\partial \gamma_2} = -\frac{1}{2} \operatorname{trace}(H^{-1} S_2) + \frac{1}{2\sigma_1^2} (W - AX)^{T} H^{-1} S_2 H^{-1} (W - AX) = 0$$

or, $(W - AX)^{T} H^{-1} S_2 H^{-1} (W - AX) = \sigma_1^2 \operatorname{trace}(H^{-1} S_2)$. (7.7)

If in the above equation $X(\gamma_2)$ and $\sigma_1^2(\gamma_2)$ are replaced by their respective values from equations (7.3) and (7.4), an equation is obtained with γ_2 as the only unknown.

If it is known a priorily which observations set is less reliable, one can always set $0 < \gamma_2 < 1$, and an approximate solution can be obtained by iteration on equation (7.7).

7.1.3 Solution with Only one Updated Variance-covariance Matrix

Consider a least squares adjustment of the linear model $W = AX + BV$ where $E(V) = 0$, $\operatorname{Var}(V) = \Sigma$, the well known solution of which is:

$$\hat{X} = (A^{T} M^{-1} A)^{-1} A^{T} M^{-1} W$$

$$\operatorname{Var} \hat{X} = \frac{V^{T} \Sigma^{-1} V}{f} (A^{T} M^{-1} A)^{-1}$$

where f is the degrees of freedom and $M = B \Sigma^{-1} B^{T}$.

Consider the same adjustment, but with $\operatorname{Var}(V) = \tilde{\Sigma} = k^2 \Sigma$, where k^2 is a scalar, then

$$\hat{\tilde{X}} = (A^{T} \tilde{M}^{-1} A)^{-1} A^{T} \tilde{M}^{-1} W ,$$

but $$\tilde{M} = B \tilde{\Sigma} B^{T} = k^2 B \Sigma B^{T} = k^2 M, \qquad \tilde{M}^{-1} = \frac{1}{k^2} M^{-1} .$$

Consequently

$$\hat{\tilde{X}} = k^2 (A^T M^{-1} A)^{-1} \left(\frac{1}{k^2} \right) A^T M^{-1} W = \hat{X}$$

and also since V is a linear transformation of X, $\tilde{V} = V$. Also

$$\text{Var}(\hat{\tilde{X}}) = \frac{V^T \tilde{\Sigma}^{-1} V}{f} (A^T \tilde{M}^{-1} A)^{-1} = \frac{\frac{1}{k^2} V^T \Sigma^{-1} V}{f} k^2 (A^T M^{-1} A) =$$

$$= \text{Var}(\hat{X})$$

i.e., a scalar multiplication of the variance-covariance matrix of the observations has no effect on the results.

Returning to the model with two sets of observations

$$W = AX + B_1 V_1 + B_2 V_2 = AX + [B_1 B_2] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = AX + BV$$

and

$$\text{Var}(V) = \begin{bmatrix} \text{Var}(V_1) & 0 \\ 0 & \text{Var}(V_2) \end{bmatrix} = \begin{bmatrix} \sigma_1^2 K_1 & 0 \\ 0 & \sigma_2^2 K_2 \end{bmatrix}$$

Instead of Var(V), the following variance-covariance matrix can be used without effecting the results:

$$k^2 \,\text{Var}(V) = \begin{bmatrix} k^2 \sigma_1^2 K_1 & 0 \\ 0 & k^2 \sigma_2^2 K_2 \end{bmatrix}$$

Selecting $k^2 = \frac{1}{\sigma_1^2}$, one gets

$$k^2 \,\text{Var}(V) = \begin{bmatrix} K_1 & 0 \\ 0 & \frac{\sigma_2^2}{\sigma_1^2} K_2 \end{bmatrix} = \begin{bmatrix} K_1 & 0 \\ 0 & \gamma K_2 \end{bmatrix}$$

This is exactly the same as using

$$\mathrm{Var}(V_1) = K_1$$

and $$\mathrm{Var}(V_2) = \gamma K_2$$

where only the constant γ remains to be determined. In this case

$$\mathrm{Var}(W) = B_1 K_1 B_1^T + \gamma B_2 K_2 B_2^T = S_1 + \gamma S_2 = H \ .$$

The likelihood of W is given by the same expression as before.

$$\lambda = \log L = -\frac{n}{2} \log(2\pi) - \log\sqrt{|H|} - \frac{(W-AX)^T H^{-1} (W-AX)}{2} \tag{7.8}$$

Setting $\dfrac{\partial\lambda}{\partial X} = 0$ and $\dfrac{\partial\lambda}{\partial\gamma} = 0$, one can derive that

$$(A^T H^{-1} A)X = A^T H^{-1} W \tag{7.9}$$

and $$(W-AX)^T H^{-1} \frac{\partial H}{\partial\gamma} H^{-1} (W-AX) = \mathrm{trace}\ (H^{-1}\frac{\partial H}{\partial\gamma})$$

Since $H = S_1 + \gamma S_2$, $\dfrac{\partial H}{\partial\gamma} = S_2$, so that

$$(W-AX)^T H^{-1} S_2 H^{-1} (W-AX) = \mathrm{trace}\ (H^{-1} S_2). \tag{7.10}$$

Let

$$H^{-1} S_2 H^{-1} = (H S_2^{-1} H)^{-1} = \Omega^{-1}$$

with $$\Omega = H S_2^{-1} H = (S_1 + \gamma S_2) S_2^{-1} (S_1 + \gamma S_2) =$$

$$= S_1 S_2^{-1} S_1 + 2\gamma S_1 + \gamma^2 S_2 \ .$$

Also $H^{-1} S_2 = [S_2^{-1}(S_1 + \gamma S_2)]^{-1} = (S_2^{-1} S_1 + \gamma I)^{-1}$. Substituting the above equations into (7.10), one arrives at the equation which is to be solved for γ:

$$(W-AX)^T \Omega^{-1} (W-AX) = \mathrm{trace}\ [(S_2^{-1} S_1 + \gamma I)^{-1}] \tag{7.11}$$

where $$X = (A^T H^{-1} A)^{-1} A^T H^{-1} W$$

and

$$\Omega = S_1 S_2^{-1} S_1 + 2\gamma S_1 + \gamma^2 S_2 .$$

If γ is needed with a low accuracy only, and the algorithm is arranged so that $0<\gamma<1$, equation (7.8) will provide λ for a given γ. The value of γ at which λ is maximum will also make the following λ' function a maximum:

$$\lambda' = -\log\sqrt{|H|} - \frac{(W - AX)^T H^{-1} (W - AX)}{2}$$

Thus computing λ' for various values of γ, at some equal increments (say, for example, $\gamma = 0.1k$, with $k = 1, 2, \cdots 10$), a graph of $\lambda' = \lambda'(\gamma)$ can be constructed, which at its maximum will provide the sought γ.

7.2 The MINQUE Criterion

The criterion for optimal weighting in this case is the minimization of an appropriate norm, established by C. R. Rao in his so called MINQUE theory (Minimum Norm Quadratic Unbiased Estimator). The theory is developed in [15], chapter 4j, and in a series of papers ([16], [17] and [18]). Only a very short outline of the theory is given here. Rao considers the linear model

$$Y = X\beta + \epsilon$$

where $\epsilon = U_1\xi_1 + U_2\xi_2 + \cdots U_k\xi_k = U\xi$ in which

$$E\left\{\xi_i\right\} = 0 \quad \text{and} \quad E\left\{\xi_i \xi_j^T\right\} = \sigma_i^2 \delta_{ij} I \quad (\sigma_i^2 \text{ are unknown})$$

$$E(\epsilon\epsilon^T) = \sigma_1^2 U_1 U_1^T + \cdots \sigma_k^2 U_k U_k^T = \sigma_1^2 V_1 + \cdots \sigma_k^2 V_k ,$$

where Y are the known observations, X is a known matrix and β is a vector of parameters to be estimated. The quantity to be estimated is a linear combination of the σ_i^2

$$\sum_i p_i \sigma_i^2 = p^T \sigma$$

where $p^T = [p_1, p_2, \cdots p_k]$ and $\sigma^T = [\sigma_1^2, \sigma_2^2, \cdots \sigma_k^2]$. The proposed estimator is a quadratic form $Y^T AY$, where the matrix A remains to be determined. The estimator is also required to have some additional properties which impose

additional constraints on the matrix A. These properties are the following.

a.) Invariance. If $\beta' = \beta - \beta_0$, where β_0 is a constant vector, the following relationship needs to be fulfilled:

$$Y'^T A Y' = Y^T A Y$$

where $Y' = X\beta' + \epsilon$. This property imposes the condition that $AX = 0$.

b.) Unbiased. The following relationship needs to be fulfilled:

$$E\left\{Y^T A Y\right\} = \sum_i p_i \sigma_i^2 = p^T \sigma .$$

The corresponding condition is that $\mathrm{Trace}(AV_i) = p_i$.

c.) Minimum Norm. If the ξ_i's were known, a natural estimator would be

$$\sigma_i^2 = \frac{\xi_i^T \xi_i}{n_i}$$

where n_i = number of elements in ξ_i. Then $\sum_i p_i \sigma_i^2 = \sum_i \frac{p_i}{n_i} \xi_i^T \xi_i = \xi^T \Delta \xi$, where $\xi^T = [\xi_1^T, \xi_2^T, \cdots \xi_k^T]$ and $\Delta_{ij} = \delta_{ij} \frac{p_i}{n_i}$.

The difference of the estimators is

$$Y^T A Y - \xi^T \Delta \xi$$

from where, using the condition $AX = 0$, one gets

$$Y^T A Y - \xi^T \Delta \xi = \xi^T U^T A U \xi - \xi^T \Delta \xi = \xi^T (U^T A U - \Delta) \xi \; :$$

It is desired to have the norm $\| U^T A U - \Delta \| = \min.$ for some properly defined norm.

7.2.1 Solution Under Euclidean Norm

Consider the Euclidean norm defined as $\|B\|^2 = \sum_i \sum_j (B_{ij})^2$, then the norm to be minimized will become

$$\| U^T A U - \Delta \| = \mathrm{Tr}(AVAV) - \mathrm{Tr}(\Delta\Delta)$$

where

$$V = \sum_{i=1}^{k} V_i = \sum_i U_i U_i^T .$$

Since Δ is fixed, the problem reduces to minimizing $\mathrm{Tr}(AVAV)$, subject to the conditions $AX = 0$ and $\mathrm{Tr}(AV_i) = p_i$. The solution given by Rao is as follows:

$$A = \sum_i \lambda_i RV_i R$$

where $R = V^{-1} - V^{-1}X(X^T V^{-1} X)^{-1} X^T V^{-1}$ and λ_i is the solution of

$$\sum_i \lambda_i \mathrm{Tr}(RV_i RV_j) = p_j$$

or $$S\lambda = p,$$

where $\lambda^T = [\lambda_1, \lambda_2, \cdots \lambda_k]$ and $S_{ij} = \mathrm{Tr}(RV_i RV_j)$.

For the estimated quantity

$$p^T \hat{\sigma} = \sum_i p_i \hat{\sigma}_i^2 = Y^T AY = \lambda^T Q,$$

where $Q = [Q_1 \mid Q_2 \mid \cdots Q_k]$ and $Q_i = Y^T RV_i RY$. Combining $S\lambda = p$ and $p^T\hat{\sigma} = \lambda^T Q$, one gets

$$\hat{\sigma} = S^{-1} Q$$

where again $S_{ij} = \mathrm{Tr}(RV_i RV_j)$ and $Q_i = Y^T RV_i RY$.

7.2.2 Application of the MINQUE Theory to the Optimal Weighting Problem

The model is (see equations following (7.2)):

$$W = AX + B_1 V_1 + B_2 V_2 = AX + e$$

$$E(V_1) = E(V_2) = 0 \qquad E(V_1 V_2^T) = E(V_2 V_1^T) = 0$$

$$E(V_1 V_1^T) = \sigma_1^2 K_1 \qquad E(V_2 V_2^T) = \sigma_2^2 K_2$$

$$\mathrm{Var}(W) = E(ee^T) = B_1 E(V_1 V_1^T) B_1^T + B_2 E(V_2 V_2^T) B_2^T =$$

$$= \sigma_1^2 \; B_1 K_1 B_1^T + \sigma_2^2 \; B_2 K_2 B_2^T =$$

$$= \sigma_1^2 \; \Sigma_1 + \sigma_2^2 \, \Sigma_2 .$$

The MINQUE solution for σ_1^2 and σ_2^2 is

$$\begin{bmatrix} \sigma_1^2 \\ \sigma_2^2 \end{bmatrix} = S^{-1} q$$

where $S_{ij} = \mathrm{Tr}(R\Sigma_i R\Sigma_j)$, $q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ with

$$q_i = W^T R\Sigma_i RW \quad , \quad i = 1,\ 2$$

$$R = H^{-1} - H^{-1}A(A^T H^{-1}A)^{-1}A^T H^{-1}$$

$$H = \Sigma_1 + \Sigma_2 \ .$$

The computational algorithm is summarized below:

$$\Sigma_1 = B_1 K_1 B_1^T$$

$$\Sigma_2 = B_2 K_2 B_2^T$$

$$H = \Sigma_1 + \Sigma_2$$

$$R = H^{-1}[I - A(A^T H^{-1}A)^{-1}A^T H^{-1}]$$

$$q_1 = W^T R\Sigma_1 RW$$

$$q_2 = W^T R\Sigma_2 RW$$

$$S_{11} = \mathrm{Trace}(R\Sigma_1 R\Sigma_1)$$

$$S_{12} = \mathrm{Trace}(R\Sigma_1 R\Sigma_2)$$

$$S_{21} = \mathrm{Trace}(R\Sigma_2 R\Sigma_1)$$

$$S_{22} = \mathrm{Trace}(R\Sigma_2 R\Sigma_2)$$

and

$$\begin{bmatrix} \sigma_1^2 \\ \sigma_2^2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}^{-1} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$D = \sigma_1^2\Sigma_1 + \sigma_2^2\Sigma_2 \qquad \text{and} \qquad \hat{X} = (A^T D^{-1} A)^{-1} A^T D^{-1} W.$$

The main computational task is in the computation of R, but the computations are of the same order as those involved in a normal least squares solution when σ_1^2 and σ_2^2 are known.

8. SUMMARY AND CONCLUSIONS

This work consists of two problems in the establishment of selenodetic control: the determination of the shape and scale of a network of points on the lunar surface and the determination of the relationship between this network and a physically meaningful selenodetic system, namely that defined by the selenocentric principal axes of inertia. The main source of shape information is lunar terrain photography. Supporting altimetry and stellar photography, beyond the obvious scaling and orientation, also contribute significantly to the strength of the network. The role of coordinate systems, especially those inherent in the external information (lunar theory, physical libration theory), has been analyzed and it has been shown that positioning and orientation of the network is possible only with respect to a.) the estimated "selenocenter" provided by the lunar theory used in the orbital analysis and b.) the estimated "principal axes of inertia," provided by the libration theory used in both orbital analysis and the reduction of stellar photography. The possibility of orientation of the network with respect to the principal axes of inertia system by means of gravity information has been shown to be impossible in view of the present limited accuracy of second degree harmonics of the gravity field of the moon. In view of indications in previous work of inconsistencies in the estimates and statistics of altimetry, stellar photography and orbital support data, statistical tests have been described for the recovery of such inconsistencies.

The role of minimal constrained soltuions has been emphasized in retaining coordinates as parameters, although the observations involved in solutions and statistical tests may not provide information on positioning or orientation or scale.

An algorithm has been developed for the use of inner constraints, as a computationally optimum set of minimal constraints. With special reference to the orbital support, the possibility of inconsistencies in system definition between different passes, has been elaborated. Procedures for statistically establishing such inconsistencies have been outlined and the idea of the utilization of such passes

as free of positioning and orientation has been introduced as a means of reducing the effects of their inconsistencies.

Finally, in the belief that inconsistencies in orbital support data are partly due to non-realistic statistics, arising from the uncertainties in lunar gravity (on which orbital analysis heavily depends), methods have been introduced for the optimal weighting of orbital support versus terrain photography. No computational effort using real data has been made, however, towards the use of the outlined techniques. It is therefore proposed that the following numerical investigations be undertaken:

1.) Lunar photography with orbital support data should be used in a solution with respect to an arbitrary system, established by minimal constraints. In this solution each orbit pass should be treated free of position and orientation, and the estimates of shifts and orientation angles of each orbit pass and their variance-covariance matrices should be tested for statistical significance.

2.) A second solution should involve lunar photography and orbital support data, where only those passes with significant inconsistencies are treated as coordinate system free. This solution should be compared with solutions in which altimetry and stellar photography data are included, and the results should be tested for inconsistencies in those latest data sets.

3.) If no inconsistencies have been found in altimetry and stellar photography, or if they have been removed, a final solution would involve all data types, where orbital support is optimally weighted versus the lunar photography data, for the removal of any remaining inconsistencies.

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Appendix A (Referenced in section 6.2.5)

INNER CONSTRAINTS IN PHOTOGRAMMETRY

Consider the photogrammetric problem: $L = f(X_1, X_2)$, where L are photo coordinates of a cluster of points; X_1 are the unknown coordinates of those points and X_2 are the unknown elements of the exterior (and possibly interior) orientation of the camera.

The application of a least squares adjustment leads to a set of normal equations

$$NX + U = O$$

where the matrix N is singular with rank deficiency seven, due to the lack of orientation positioning and scale in our model. If N is a $u \times u$ matrix, then one way of obtaining a solution for X is to introduce a set of minimal constraints ${}_7C_u X_1 = {}_7O_1$, such that

$$\det \begin{bmatrix} N & E^T \\ C & D \end{bmatrix} \neq O .$$

Among all possible C matrices there exists some matrix E such that if

$$\begin{bmatrix} N & E^T \\ E & O \end{bmatrix}^{-1} = \begin{bmatrix} Q & M^T \\ M & L \end{bmatrix}$$

then Q is the pseudo-inverse of N, i.e., a matrix fulfilling the four relations:

$$NQN = N$$
$$QNQ = Q$$
$$NQ = (NQ)^T$$
$$QN = (QN)^T .$$

The set of constraints $EX = O$ is then said to be a set of inner constraints.

It can be shown that the matrix E also fulfills the relations

$$\det(EE^T) \neq O \quad \text{and} \quad AE^T = O$$

or, since $N = A^T PA$, the equivalent $NE^T = O$.

The first step in the investigation to determine the matrix E was to follow the procedure in [2]. According to this procedure, the matrix E is formed as the matrix of the partials of the parameters X, with respect to differential translations $d\sigma_X$, $d\sigma_Y$, $d\sigma_Z$, differential rotations $d\varphi_1$, $d\varphi_2$, $d\varphi_3$ and differential scaling $d\epsilon$ of the coordinate system. The results showed that the changes in the angles of camera orientation were non-linear functions of the differential rotation components. Following the approximations in [2], page 18,

$$\cos d\varphi_i \approx 1, \quad \sin d\varphi_i \approx d\varphi_i, \quad d\varphi_i = d\varphi_j = 0$$

a linear relation is obtained but the rsulting E matrix does not fulfill the relation $AE^T = 0$. A more careful investigation showed that the above approximations are not valid and that Blaha's results, which are correct despite the approximations, can be derived rigorously.

The question that posed itself next, was whether or not a set of constraints that does not involve the camera orientation parameters can be an inner set of constraints. Augmenting the normal equations

$$\begin{bmatrix} N_{11} & N_{12} \\ N_{12}^T & N_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = 0$$

in such a way that X_1 are the coordinates of the control network and X_2 the parameters of camera orientation, a set of constraints involving only X_1 can be written as follows:

$$E_1 X_1 = 0 \quad \text{or} \quad EX = [\, E_1 \;\; 0 \,] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Recognizing that $N = A^T PA$ or for simplicity and without loss of generality $N = A^T A$, one gets

$$N = A^T A = \begin{bmatrix} A_1^T \\ A_2^T \end{bmatrix} [A_1 \;\; A_2] = \begin{bmatrix} A_1^T A_1 & A_1^T A_2 \\ A_2^T A_2 & A_2^T A_2 \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{12}^T & N_{22} \end{bmatrix}.$$

Now select a matrix E_1 such that

$$\det (E_1 E_1^T) \neq 0 \quad \text{and} \quad N_{11} E_1^T = 0 \quad \text{or} \quad A_1 E_1^T) = 0$$

Then the matrix E fulfills the relations

$$EE^T = [E_1 \;\; 0] \begin{bmatrix} E_1^T \\ 0 \end{bmatrix} = E_1 E_1^T \qquad \det\,(EE^T) = \det\,(E_1 E_1^T \neq 0$$

$$NE^T = \begin{bmatrix} N_{11} & N_{12} \\ N_{12}^T & N_{22} \end{bmatrix} \begin{bmatrix} E_1^T \\ 0 \end{bmatrix} = \begin{bmatrix} N_{11} E_1^T \\ N_{12}^T E_1^T \end{bmatrix} = \begin{bmatrix} A_1^T A_1 E_1^T \\ A_2^T A_1 E_1^T \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \;,$$

and therefore EX=0 is an inner set of constraints.

The constrained normal equations now become:

$$\begin{bmatrix} N_{11} & N_{12} & E_1^T \\ N_{12} & N_{22} & 0 \\ E_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ K_c \end{bmatrix} + \begin{bmatrix} U_1 \\ U_2 \\ 0 \end{bmatrix} = 0$$

where K_c is the vector of the Lagrange multipliers. If the order of equations and unknowns is changed

$$\begin{bmatrix} N_{11} & E_1^T & N_{12} \\ E_1 & 0 & 0 \\ N_{12}^T & 0 & N_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ K_c \\ X_2 \end{bmatrix} + \begin{bmatrix} U_1 \\ 0 \\ U_2 \end{bmatrix} = 0$$

Since N is singular both N_{11} and N_{22} must be singular, because otherwise N could be inverted by partitioning utilizing the inverse of the non-singular matrix among N_{11} and N_{22}. Now it is possible to invert by partitioning,

$$\left[\begin{array}{cc|c} N_{11} & E_1^T & N_{12} \\ E_1 & 0 & 0 \\ \hline N_{12}^T & 0 & N_{22} \end{array}\right] \qquad \text{since} \qquad \begin{bmatrix} N_{11} & E_1^T \\ E_1 & 0 \end{bmatrix} \qquad \text{is not singular.}$$

The remaining question is how to find the matrix E_1 in such a form that $A_1 \;E_1^T = 0$ $(N_{11} \;E_1^T = 0)$ and $\det\,(E_1 \;E_1^T) \neq 0$.

If N_{11} is an nxn matrix it can be augmented

$$\underset{n \times n}{N_{11}} = \begin{bmatrix} \underset{n-7 \,\times\, n-7}{N_{111}} & \underset{n-7 \,\times\, 7}{N_{112}} \\ \underset{7 \,\times\, n-7}{N_{112}^T} & \underset{7 \,\times\, 7}{N_{122}} \end{bmatrix}$$

in such a way (by rearranging the order of unknowns if necessary) that N_{111} is a non-singular matrix and $N_{122} = N_{112}^T N_{111}^{-1} N_{112}$.

Setting also $\underset{F \times N}{E_1} = [\underset{F \times N-F}{E_{11}} \quad \underset{F \times F}{E_{12}}]$ one must have

$$N_{11}\ E_1 = \begin{bmatrix} N_{111} & N_{112} \\ N_{112} & N_{122} \end{bmatrix} \cdot \begin{bmatrix} E_{11}^T \\ E_{12}^T \end{bmatrix} = 0$$

or $\quad N_{111}\ E_{11}^T + N_{112}\ E_{12}^T = 0$

$$N_{112}^T\ E_{11}^T + N_{122}\ E_{12}^T = 0$$

If $E_{12}^T = I$, the first of the above equations gives

$$E_{11}^T = -N_{111}^{-1}\ N_{112}$$

and setting this value into the second equation one obtains

$$N_{122} = N_{112}^T\ N_{111}^{-1}\ N_{112}$$

which relation is known to hold a priorily.

Therefore,

$$E_1^T = \begin{bmatrix} E_{11}^T \\ E_{12}^T \end{bmatrix} = \begin{bmatrix} -N_{111}^{-1}\ N_{112} \\ I \end{bmatrix}$$

A different approach will be to find the matrix E_1 in analytical form with some sort of systematic pattern, such that $A_1 E_1^T = 0$ and $\det(E_1 E_1^T) \neq 0$. However, such an approach will not reduce the total computational effort, since to obtain a solution the matrix N_{111} will have to be inverted explicitly or implicitly.

Now the solution to the system becomes

$$\begin{bmatrix} X_1 \\ K_c \\ X_2 \end{bmatrix} = -\begin{bmatrix} N_{11} & E_1^T & N_{12} \\ E_1 & 0 & 0 \\ N_{12}^T & 0 & N_{22} \end{bmatrix}^{-1} \begin{bmatrix} U_1 \\ 0 \\ U_2 \end{bmatrix} = -\begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12}^T & Q_{22} & Q_{23} \\ Q_{13}^T & Q_{23}^T & Q_{33} \end{bmatrix} \begin{bmatrix} U_1 \\ 0 \\ U_2 \end{bmatrix}$$

If only the parameters X_1 are of interest, then

$$X_1 = -Q_{11}\ U_1 - Q_{13}\ U_2$$

Setting

$$\begin{bmatrix} N_{11} & E_1^T \\ E_1 & 0 \end{bmatrix}^{-1} = S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}$$

and inverting by partitioning one obtains

$$Q_{33} = \left\{ N_{22} - [\, N_{12}^{T} \; 0 \,] \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{T} & S_{22} \end{bmatrix} \begin{bmatrix} N_{12} \\ 0 \end{bmatrix} \right\}^{-1}$$

$$Q_{33} = (N_{22} - N_{12}^{T} S_{11} \; N_{12})^{-1}$$

$$\begin{bmatrix} Q_{13} \\ Q_{23} \end{bmatrix} = - \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{T} & S_{22} \end{bmatrix} \begin{bmatrix} N_{12} \\ 0 \end{bmatrix} Q_{33}$$

$$Q_{13} = - S_{11} \; N_{12} \; Q_{33}$$

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^{T} & Q_{22} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{T} & S_{22} \end{bmatrix} + \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{T} & S_{11} \end{bmatrix} \begin{bmatrix} N_{12} \\ 0 \end{bmatrix} Q_{33} \; [\, N_{12}^{T} \; 0 \,] \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{T} & S_{11} \end{bmatrix}$$

$$Q_{11} = S_{11} + S_{11} \; N_{12} \; Q_{33} \; N_{12}^{T} \; S_{11}$$

S_{11} can be found by inverting the matrix $\begin{bmatrix} N_{11} & E_1^{T} \\ E_1 & 0 \end{bmatrix}$.

Another approach is to recognize the fact that S_{11} is the pseudoinverse of N_{11} ($S_{11} = N_{11}^{+}$) and to use some direct algorithm to find this pseudoinverse.

Appendix B (Referenced in section 5.1)

COMPUTATION OF VARIANCE-COVARIANCE MATRICES OF CAMERA ORIENTATION ANGLES WITH RESPECT TO A MOON-FIXED SYSTEM

If K_j , Φ_j , Ω_j are the orientation angles at j^{th} camera exposure with respect to an inertial system, and ψ_j , θ_j , ϵ_j are the Eulerian angles relating a moon-fixed system to the inertial one, then the relationship between the orientation angles of the camera (κ_j , φ_j , ω_j) with respect to the moon-fixed system is

$$M(\kappa_j, \varphi_j, \omega_j) = \widetilde{M}(K_j, \Phi_j, \Omega_j)\,\widetilde{R}(\psi_j, \theta_j, \epsilon_j)$$

where

$$M = R_3(\kappa_j)\,R_2(\varphi_j)\,R_1(\omega_j)$$

$$\widetilde{M} = R_3(K_j)\,R_2(\Phi_j)\,R_1(\Omega_j)$$

$$\widetilde{R} = R_3(-\epsilon_j)\,R_1(\theta_j)\,R_3(-\psi_j)\ .$$

Introducing the notation

$$E = [\,\kappa_j\ \varphi_j\ \omega_j\,]^T$$

$$F = [\,K_j\ \Phi_j\ \Omega_j\,]^T$$

$$G = [\,\epsilon_j\ \theta_j\ \psi_j\,]^T\ ,$$

the corresponding variance-covariance matrices are given by

$$\Sigma_E = Q_F\,\Sigma_F\,Q_F^T + Q_G\,\Sigma_G\,Q_G^T\ ,$$

where $Q_F = \dfrac{\partial E}{\partial F}$ and $Q_G = \dfrac{\partial E}{\partial G}$.

If m_{ij} are the elements of the matrix M, one gets

$$\kappa_j = \tan^{-1}\left(\frac{-m_{21}}{m_{11}}\right),\quad \varphi_j = \tan^{-1}\left(\frac{m_{31}}{\sqrt{m_{32}^2 + m_{33}^2}}\right),$$

$$\omega_j = \tan^{-1}\left(\frac{-m_{31}}{m_{33}}\right)$$

and:

$$\frac{\partial E}{\partial a} = \begin{bmatrix} m_{21} & -m_{11} & 0 & 0 & 0 \\ 0 & 0 & \cos\varphi_J & -\tan\varphi_J & -\tan\varphi_J \\ 0 & 0 & -m_{33} & 0 & m_{31} \end{bmatrix} \frac{\partial}{\partial a} \begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \\ m_{32} \\ m_{33} \end{bmatrix}$$

There is also a need to compute $\partial M/\partial a$, where a is an element of F or G.

$$\frac{\partial M}{\partial a_F} = \frac{\partial \widetilde{M}}{\partial a_F}\widetilde{R} \qquad \text{and} \qquad \frac{\partial M}{\partial a_G} = \widetilde{M}\frac{\partial \widetilde{R}}{\partial a_G},$$

where a_F is a component of F, and a_G is a component of G.

Using the matrices

$$P_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

one gets the following relations:

$$\widetilde{M} = R_3(K)R_2(\Phi)R_1(\Omega) \qquad \widetilde{R} = R_3(-\epsilon)R_1(\theta)R_3(-\psi)$$

$$\frac{\partial}{\partial K}\widetilde{M} = P_3 M \qquad \frac{\partial}{\partial \epsilon}\widetilde{R} = -P_3\widetilde{R}$$

$$\frac{\partial}{\partial \Phi}\widetilde{M} = R_3(K)P_2\, R_2(\Phi)R_1(\Omega) \qquad \frac{\partial}{\partial \theta}\widetilde{R} = R_3(-\epsilon)P_1\, R_1(\theta)R_3(-\psi)$$

$$\frac{\partial}{\partial \Omega}\widetilde{M} = \widetilde{M}P_1 \qquad \frac{\partial}{\partial \psi}\widetilde{R} = -\widetilde{R}P_3\,.$$

Appendix C (Referenced in section 3.2)

ON THE DETERMINATION OF THE DIRECTION OF THE PRINCIPAL AXES OF INERTIA FROM GRAVITY FIELD INFORMATION

1. Transformations of Moments and Products of Inertia due to Changes in the Coordinate System

Assume only Cartesian systems with origin at the center of mass of the body in question. If $0, x, y, z$ and $0, \bar{x}, \bar{y}, \bar{z}$ are two such systems with corresponding moments and products of inertia, A, B, C, D, E, F and $\bar{A}, \bar{B}, \bar{C}, \bar{D}, \bar{E}, \bar{F}$, then the transformation from one system to the other can be represented as:

$$\begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} = R_3(\varkappa) R_2(\varphi) R_1(\omega) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The problem is to find $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$, $\bar{E}$, $\bar{F}$ given A, B, C, D, E, F and the angles $\varkappa$, φ, ω. Introduce the following auxiliary coordinate systems:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R_1(\omega) \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = R_2(\varphi) \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

$$\begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} = R_3(\varkappa) \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix}$$

The corresponding moments and products of inertia are

$$A' = \int_M (z'^2 + y'^2)dm =$$

$$= \int_M [(\cos\omega y + \sin\omega z)^2 + (-\sin\omega y + \cos\omega z)^2]dm =$$

$$= \int_M (y^2 + z^2)\,dm = A$$

$$B' = \int_M (x'^2 + z'^2)\,dm = \int_M [x^2 + (-\sin\omega y + \cos\omega z)^2]dm =$$

$$= \sin^2\omega \int_M (x^2 + y^2)dm + \cos^2\omega \int (x^2 + z^2)dm -$$

$$2\cos\omega \sin\omega \int_M yz\,dm =$$

$$= \sin^2\omega\ C + \cos^2\omega\ B - \sin 2\omega\, D$$

$$C' = \int_M (x'^2 + y'^2)dm = \int_M [x^2 + (\cos\omega y + \sin\omega z)^2]dm =$$

$$= \cos^2\omega \int_M (x^2 + y^2)dm + \sin^2\omega \int (x^2 + z^2)dm +$$

$$2\cos\omega \sin\omega \int_M yz\,dm$$

$$= \cos^2\omega\, C + \sin^2\omega\, B + \sin 2\omega\, D$$

$$D' = \int_M y'z'\,dm = \int_M (\cos\omega y + \sin\omega z)\,(-\sin\omega y + \cos\omega z)dm =$$

$$= \sin\omega \cos\omega \left[\int_M z^2\,dm - \int_M y^2\,dm \right] +$$

$$(\cos^2\omega - \sin^2\omega) \int_M yz\,dm =$$

$$= \sin\omega \cos\omega \left[\frac{A+B-C}{2} - \frac{A-B+C}{2} \right] + \cos 2\omega\, D$$

$$= \sin\omega \cos\omega\, (B-C) + \cos 2\omega\, D$$

$$E' = \int_M x'z'\, dm = \int_M x(-\sin\omega y + \cos\omega z) dm =$$

$$= -\sin\omega \int_M xy\, dm + \cos\omega \int_M xz\, dm = -\sin\omega\, F + \cos\omega\, E$$

$$F' = \int_M x'y'\, dm = \int_M x(\cos\omega\, y + \sin\omega\, z) dm =$$

$$= \cos\omega\, F + \sin\omega\, E\ .$$

In summary

$$\begin{aligned}
A' &= A \\
B' &= \cos^2\omega\, B + \sin^2\omega\, C - \sin 2\omega\, D \\
C' &= \sin^2\omega\, B + \cos^2\omega\, C + \sin 2\omega\, D \\
D' &= \frac{1}{2}\sin 2\omega\, (B-C) + \sin 2\omega\, D \\
E' &= \cos\omega\, E - \sin\omega \cdot F \\
F' &= \cos\omega\, F + \sin\omega\, E\ .
\end{aligned} \tag{C.1a}$$

Similarly

$$\begin{aligned}
A'' &= \cos^2\varphi\, A' + \sin^2\varphi\, C' + \sin 2\varphi\, E' \\
B'' &= B' \\
C'' &= \sin^2\varphi\, A' + \cos^2\varphi\, C' - \sin 2\varphi\, E' \\
D'' &= \cos\varphi\, D' + \sin\varphi\, F' \\
E'' &= \frac{1}{2}\sin 2\varphi\, (C' - A') + \cos 2\varphi\, E' \\
F'' &= -\sin\varphi\, D' + \cos\varphi\, F'
\end{aligned} \tag{C.1b}$$

and

$$
\begin{aligned}
\overline{A} &= \cos^2\varkappa\, A'' + \sin^2\varkappa\, B'' - \sin 2\varkappa\, F'' \\
\overline{B} &= \sin^2\varkappa\, A'' + \cos^2\varkappa\, B'' + \sin 2\varkappa\, F'' \\
\overline{C} &= C'' \\
\overline{D} &= \cos\varkappa\, D'' - \sin\varkappa\, E'' \\
\overline{E} &= \sin\varkappa\, D'' + \cos\varkappa\, E'' \\
\overline{F} &= \frac{1}{2}\sin 2\varkappa\,(A'' - B'') + \cos 2\varkappa\, F'' .
\end{aligned}
\tag{C.1c}
$$

Combining the above three sets of equations, the following final transformation equations are obtained:

$$
\begin{aligned}
\overline{A} = {} & A\left\{\cos^2\varkappa\,\cos^2\varphi\right\} \\
& + B\left\{\sin^2\varkappa\,\cos^2\omega + \cos^2\varkappa\,\sin^2\varphi\,\sin^2\omega + \frac{1}{2}\sin 2\varkappa\,\sin\varphi\,\sin 2\omega\right\} \\
& + C\left\{\sin^2\varkappa\,\sin^2\omega + \cos^2\varkappa\,\sin^2\varphi\,\cos^2\omega - \frac{1}{2}\sin 2\varkappa\,\sin\varphi\,\sin 2\omega\right\} \\
& + D\left\{-\sin^2\varkappa\,\sin 2\omega + \cos^2\varkappa\,\sin^2\varphi\,\sin 2\omega + \sin 2\varkappa\,\sin\varphi\,\cos 2\omega\right\} \\
& + E\left\{\cos^2\varkappa\,\sin 2\varphi\,\cos\omega - \sin 2\varkappa\,\cos\varphi\,\sin\omega\right\} \\
& + F\left\{-\cos^2\varkappa\,\sin 2\varphi\,\sin\omega - \sin 2\varkappa\,\cos\varphi\,\cos\omega\right\}
\end{aligned}
\tag{C. 2a}
$$

$$
\begin{aligned}
\overline{B} = {} & A\left\{\sin^2\varkappa\,\cos^2\varphi\right\} \\
& + B\left\{\cos^2\varkappa\,\cos^2\omega + \sin^2\varkappa\,\sin^2\varphi\,\sin^2\omega - \frac{1}{2}\sin 2\varkappa\,\sin\varphi\,\sin 2\omega\right\} \\
& + C\left\{\cos^2\varkappa\,\sin^2\omega + \sin^2\varkappa\,\sin^2\varphi\,\cos^2\omega + \frac{1}{2}\sin 2\varkappa\,\sin\varphi\,\sin 2\omega\right\} \\
& + D\left\{-\cos^2\varkappa\,\sin 2\omega + \sin^2\varkappa\,\sin^2\varphi\,\sin 2\omega - \sin 2\varkappa\,\sin\varphi\,\cos 2\omega\right\} \\
& + E\left\{\sin^2\varkappa\,\sin 2\varphi\,\cos\omega + \sin 2\varkappa\,\cos\varphi\,\sin\omega\right\} \\
& + F\left\{-\sin^2\varkappa\,\sin 2\varphi\,\sin\omega + \sin 2\varkappa\,\cos\varphi\,\cos\omega\right\}
\end{aligned}
\tag{C. 2a}
$$

$$
\begin{aligned}
\overline{C} = {} & A\left\{\sin^2\varphi\right\} \\
& + B\left\{\cos^2\varphi\,\sin^2\omega\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \; C\left\{\cos^2\varphi \;\cos^2\omega\right\} \\
&+ \; D\left\{\cos^2\varphi \;\sin 2\omega\right\} \\
&+ \; E\left\{-\sin 2\varphi \;\cos\omega\right\} \\
&+ \; F\left\{\sin 2\varphi \;\sin\omega\right\}
\end{aligned}
\tag{C. 2a}
$$

$$
\begin{aligned}
\overline{D} \;=\; & A\left\{\sin\varkappa \;\cos\varphi \;\sin\varphi\right\} \\
&+ B\left\{-\frac{1}{2}\sin\varkappa \;\sin 2\varphi \;\sin^2\omega + \frac{1}{2}\cos\varkappa \;\cos\varphi \;\sin 2\omega\right\} \\
&+ C\left\{-\frac{1}{2}\sin\varkappa \;\sin 2\varphi \;\cos^2\omega - \frac{1}{2}\cos\varkappa \;\cos\varphi \;\sin 2\omega\right\} \\
&+ D\left\{-\frac{1}{2}\sin\varkappa \;\sin 2\varphi \;\sin 2\omega + \cos\varkappa \;\cos\varphi \;\cos 2\omega\right\} \\
&+ E\left\{-\sin\varkappa \;\cos 2\varphi \;\cos\omega + \cos\varkappa \;\sin\varphi \;\sin\omega\right\} \\
&+ F\left\{\sin\varkappa \;\cos 2\varphi \;\sin\omega + \cos\varkappa \;\sin\varphi \;\cos\omega\right\}
\end{aligned}
\tag{C. 2b}
$$

$$
\begin{aligned}
\overline{E} \;=\; & A\left\{-\cos\varkappa \;\cos\varphi \;\sin\varphi\right\} \\
&+ B\left\{\frac{1}{2}\cos\varkappa \;\sin 2\varphi \;\sin^2\omega + \frac{1}{2}\sin\varkappa \;\cos\varphi \;\sin 2\omega\right\} \\
&+ C\left\{\frac{1}{2}\cos\varkappa \;\sin 2\varphi \;\cos^2\omega - \frac{1}{2}\sin\varkappa \;\cos\varphi \;\sin 2\omega\right\} \\
&+ D\left\{\frac{1}{2}\cos\varkappa \;\sin 2\varphi \;\sin 2\omega + \sin\varkappa \;\cos\varphi \;\cos 2\omega\right\} \\
&+ E\left\{\cos\varkappa \;\cos 2\varphi \;\cos\omega + \sin\varkappa \;\sin\varphi \;\sin\omega\right\} \\
&+ F\left\{-\cos\varkappa \;\cos 2\varphi \;\sin\omega + \sin\varkappa \;\sin\varphi \;\cos\omega\right\}
\end{aligned}
\tag{C. 2b}
$$

$$
\begin{aligned}
\overline{F} \;=\; & A\left\{\cos\varkappa \;\sin\varkappa \;\cos^2\varphi\right\} \\
&+ B\left\{\frac{1}{2}\sin 2\varkappa \;\sin^2\varphi \;\sin^2\omega - \frac{1}{2}\cos 2\varkappa \;\sin\varphi \;\sin 2\omega - \frac{1}{2}\sin 2\varkappa \;\cos^2\omega\right\} \\
&+ C\left\{\frac{1}{2}\sin 2\varkappa \;\sin^2\varphi \;\cos^2\omega + \frac{1}{2}\cos 2\varkappa \;\sin\varphi \;\sin 2\omega - \frac{1}{2}\sin 2\varkappa \;\sin^2\omega\right\} \\
&+ D\left\{\frac{1}{2}\sin 2\varkappa \;\sin^2\varphi \;\sin 2\omega - \cos 2\varkappa \;\sin\varphi \;\cos 2\omega + \frac{1}{2}\sin 2\varkappa \;\sin 2\omega\right\} \\
&+ E\left\{\frac{1}{2}\sin 2\varkappa \;\sin 2\varphi \;\cos\omega + \cos 2\varkappa \;\cos\varphi \;\sin\omega\right\} \\
&+ F\left\{-\frac{1}{2}\sin 2\varkappa \;\sin 2\varphi \;\sin\omega + \cos 2\varkappa \;\cos\varphi \;\cos\omega\right\}
\end{aligned}
\tag{C. 2b}
$$

2. Determination of the Directions of the Principal Axes of Inertia

Assume that the system x , y , z is the principal axes of inertia system, and x , y , z is an arbitrary system with the same origin for which the integrals A , B , C , D , E , F are known, then

$$\bar{D} = \bar{D}(A,B,C,D,E,F,\varkappa,\varphi,\omega) = 0$$

$$\bar{E} = \bar{E}(A,B,C,D,E,F,\varkappa,\varphi,\omega) = 0$$

$$\bar{F} = \bar{F}(A,B,C,D,E,F,\varkappa,\varphi,\omega) = 0$$

This is a system of 3 nonlinear equations which can be solved for the 3 unknowns, $\varkappa$, φ, ω. The 3 angles then, specify the orientation of the principal axes of inertia with respect to the arbitrary system x , y , z. However knowledge of all the 6 quantities A, B, C, D, E, F is not necessary. A little algebraic manipulation of the transformation equations for $\bar{D}$, $\bar{E}$ and $\bar{F}$ results in

$$\begin{aligned}
\bar{D} = {} & \frac{A-B}{2}\left\{\sin\varkappa\,\sin 2\varphi\,\sin^2\omega - \cos\varkappa\,\cos\varphi\,\sin 2\omega\right\} \\
& + \frac{A-C}{2}\left\{\sin\varkappa\,\sin 2\varphi\,\cos^2\omega + \cos\varkappa\,\cos\varphi\,\sin 2\omega\right\} \\
& + D\left\{-\frac{1}{2}\sin\varkappa\,\sin 2\varphi\,\sin 2\omega + \cos\varkappa\,\cos\varphi\,\cos 2\omega\right\} \\
& + E\left\{-\sin\varkappa\,\cos 2\varphi\,\cos\omega + \cos\varkappa\,\sin\varphi\,\sin\omega\right\} \\
& + F\left\{\sin\varkappa\,\cos 2\varphi\,\sin\omega + \cos\varkappa\,\sin\varphi\,\cos\omega\right\} = 0
\end{aligned} \tag{C.3}$$

$$\begin{aligned}
\bar{E} = {} & \frac{A-B}{2}\left\{-\cos\varkappa\,\sin 2\varphi\,\sin^2\omega - \sin\varkappa\,\cos\varphi\,\sin 2\omega\right\} \\
& + \frac{A-C}{2}\left\{-\cos\varkappa\,\sin 2\varphi\,\cos^2\omega + \sin\varkappa\,\cos\varphi\,\sin 2\omega\right\} \\
& + D\left\{\frac{1}{2}\cos\varkappa\,\sin 2\varphi\,\sin 2\omega + \sin\varkappa\,\cos\varphi\,\cos 2\omega\right\} \\
& + E\left\{\cos\varkappa\,\cos 2\varphi\,\cos\omega + \sin\varkappa\,\sin\varphi\,\sin\omega\right\}
\end{aligned}$$

$$= \quad F\left\{-\cos\varkappa\,\cos 2\varphi\,\sin\omega + \sin\varkappa\,\sin\varphi\,\cos\omega\right\} = 0 \qquad (C.3)$$

$$\begin{aligned}
\overline{F} \quad = \quad & \frac{A-B}{2}\left\{\cos 2\varkappa\,\sin\varphi\,\sin 2\omega - \sin 2\varkappa\,\sin^2\varphi\,\sin^2\omega + \sin 2\varkappa\cos^2\omega\right\} \\
+ \quad & \frac{A-C}{2}\left\{-\cos 2\varkappa\,\sin\varphi\,\sin 2\omega - \sin 2\varkappa\,\sin^2\varphi\,\cos^2\omega + \sin 2\varkappa\,\sin^2\omega\right\} \\
+ \quad & D\left\{\frac{1}{2}\,\sin 2\varkappa\,\sin^2\varphi\,\sin 2\omega - \cos 2\varkappa\,\sin\varphi\,\cos 2\omega + \frac{1}{2}\sin 2\varkappa\,\sin 2\omega\right\} \\
+ \quad & E\left\{\frac{1}{2}\sin 2\varkappa\,\sin 2\varphi\,\cos\omega + \cos 2\varkappa\,\cos\varphi\,\sin\omega\right\} \\
+ \quad & F\left\{-\frac{1}{2}\sin 2\varkappa\,\sin 2\varphi\,\sin\omega + \cos 2\varkappa\,\cos\varphi\,\cos\omega\right\} = 0\,. \qquad (C.3)
\end{aligned}$$

From the above equations one can see that if the quantities A-B, A-C, D, E, F are known, $\varkappa$, φ , ω can be computed. These quantities are related to the second degree harmonics of the gravitational field.

From [8] (p. 160, eq. 21.043) :

$$\begin{aligned}
C_{20} &= \frac{A+B}{2} - C \\
C_{21} &= E \qquad\qquad (C.4) \\
S_{21} &= D \\
C_{22} &= \frac{B-A}{4} \\
S_{22} &= \frac{1}{2}F\,,
\end{aligned}$$

where C_{nm} , S_{nm} are coefficients in an expansion of the gravitational potential of the form ([8], p. 159) ,

$$V = G\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\sum_{m=0}^{n}\left[C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right]P_{nm}(\sin\varphi)$$

r , φ , λ being spherical coordinates and G the gravitational constant.

The inverses of the above relations are

$$
\begin{aligned}
A - B &= -4C_{22} & D &= S_{21} \\
A - C &= C_{20} - 2C_{22} & E &= C_{21} \\
& & F = 2S_{22} \,.
\end{aligned}
\tag{C.5}
$$

An algorithm is still needed to solve for $\varkappa, \varphi, \omega$ in terms of C_{20}, C_{21}, C_{22}, S_{21}, S_{22}. One can always define the x, y, z system to be close to the $\bar{x}$, $\bar{y}$, $\bar{z}$. Then as a first approximation set

$$
\begin{gathered}
\cos\varkappa = \cos\varphi = \cos\omega = 1\,, \\
\sin\varkappa = \varkappa, \quad \sin\varphi = \varphi, \quad \sin\omega = \omega \quad \text{and} \\
\varkappa^2 = \varphi^2 = \omega^2 = \varkappa\varphi = \varkappa\omega = \varphi\omega = 0\,.
\end{gathered}
$$

Under these simplifications

$$
\begin{aligned}
\bar{A} &= A + 2\varphi E - 2\varkappa F \\
\bar{B} &= B - 2\omega D + 2\varkappa F \\
\bar{C} &= C + 2\omega D - 2\varphi E \\
\bar{D} &= (B - C)\omega + F\varphi - E\varkappa + D = 0 \\
\bar{E} &= -F\omega + (C - A)\varphi + D\varkappa + E = 0 \\
\bar{F} &= E\omega - D\varphi + (A - B)\varkappa + F = 0 \,.
\end{aligned}
\tag{C.6}
$$

The solution of the last three equations is

$$
\begin{bmatrix} \omega \\ \varphi \\ \varkappa \end{bmatrix} = \begin{bmatrix} C-B & -F & E \\ F & A-C & -D \\ -E & D & B-A \end{bmatrix}^{-1} \begin{bmatrix} D \\ E \\ F \end{bmatrix}
$$

And the final results are

$$\omega = \frac{D[\Delta - \alpha\beta] - EF(\alpha + \beta)}{d}$$

$$\varphi = \frac{E[\Delta + \alpha\beta - \alpha^2] + DF(2\alpha - \beta)}{d} \tag{C.7}$$

$$\kappa = \frac{F[\Delta + \alpha\beta - \beta^2] + DE(2\beta - \alpha)}{d}$$

where

$$\Delta = D^2 + E^2 + F^2$$

$$\alpha = A - B$$

$$\beta = A - C, \quad \text{and}$$

$$d = (\alpha - \beta)D^2 + \beta E^2 - \alpha F^2 - \alpha\beta(\alpha - \beta) .$$

Of course the values of κ , φ , ω are only first approximations but they can be improved by an iteration schème based on the following sets of equations:

$$\begin{aligned}
(A - B)_{i+1} = \; & (A - B)_i [\cos 2\kappa_i \cos^2\omega_i - \cos 2\kappa_i \sin^2\varphi_i \sin^2\omega_i \\
& - \sin 2\kappa_i \sin\varphi_i \sin 2\omega_i] \\
+ \; & (A - C)_i [\cos 2\kappa_i \sin^2\omega_i - \cos 2\kappa_i \sin^2\varphi_i \cos^2\omega_i \\
& + \sin 2\kappa_i \sin\varphi_i \sin 2\omega_i] \\
+ \; & D_i [\cos 2\kappa_i \sin 2\omega_i + \cos 2\kappa_i \sin^2\varphi_i \sin 2\omega_i \\
& + 2 \sin 2\kappa_i \sin\varphi_i \cos 2\omega_i] \\
+ \; & E_i [\cos 2\kappa_i \sin 2\varphi_i \cos\omega_i - 2 \sin 2\kappa_i \cos\varphi_i \sin\omega_i] \\
+ \; & F_i [- \cos 2\kappa_i \sin 2\varphi_i \sin\omega_i - 2 \sin 2\kappa_i \cos\varphi_i \cos\omega_i]
\end{aligned}$$

$$\begin{aligned}
(A - C)_{i+1} = \; & (A - B)_i [-\frac{1}{2} \sin 2\kappa_i \sin\varphi_i \sin 2\omega_i - \sin^2\kappa_i \cos^2\omega_i \\
& + \cos^2\varphi_i \sin^2\omega_i - \cos^2\kappa_i \sin^2\varphi_i \sin^2\omega_i] \\
+ \; & (A - C)_i [\frac{1}{2} \sin 2\kappa_i \sin\varphi_i \sin 2\omega_i - \sin^2\kappa_i \sin^2\omega_i
\end{aligned}$$

$$+ \cos^2\varphi_i \cos^2\omega_i - \cos^2\varkappa_i \sin^2\varphi_i \cos^2\omega_i]$$

$$+ \quad D_i[-\sin^2\varkappa_i \sin 2\omega_i + \cos^2\varkappa_i \sin^2\varphi_i \sin 2\omega_i$$

$$+ \sin 2\varkappa_i \sin\varphi_i \cos 2\omega_i - \cos^2\varphi_i \sin 2\omega_i]$$

$$+ \quad E_i[\cos^2\varkappa_i \sin 2\varphi_i \cos\omega_i - \sin 2\varkappa_i \cos\varphi_i \sin\omega_i$$

$$+ \sin 2\varphi_i \cos\omega_i]$$

$$+ \quad F_i[-\cos^2\varkappa_i \sin 2\varphi_i \sin\omega_i - \sin 2\varkappa_i \cos\varphi_i \cos\omega_i$$

$$- \sin 2\varphi_i \sin\omega_i] \ .$$

D_{i+1} , E_{i+1} , F_{i+1} can be obtained from Eq. (C.3) if in the right hand side (A - B), (A - C), D, E, F, $\varkappa$, φ, ω are replaced with $(A-B)_i$, $(A - C)_i$, D_i, E_i, F_i, $\varkappa_i$, φ_i, ω_i, respectively. Also, $\varkappa_{i+1}$, φ_{i+1}, ω_{i+1} can be obtained from Eq. (C.7) if on the right hand sides, D, E, F, (A - B), (A - C) are replaced with the same quantities with a subscript "i".

3. The Accuracy of the Directions of the Principal Axes of Inertia

The last three of equations (C.6) can be rewritten as

$$
\begin{aligned}
\bar{D} &= (C_{20} + 2C_{22})\omega + 2S_{22}\varphi - C_{21}\varkappa + S_{21} = 0 \\
\bar{E} &= -2S_{22}\omega + (2C_{22} - C_{20})\varphi + S_{21}\varkappa + C_{21} = 0 \qquad (C.8)\\
\bar{F} &= C_{21}\omega - S_{21}\varphi - 4C_{22}\varkappa + 2S_{22} = 0 .
\end{aligned}
$$

Introducing the notation

$$
\begin{aligned}
f &= [\bar{D} \;\; \bar{E} \;\; \bar{F}]^T \\
e &= [\varkappa \;\; \varphi \;\; \omega]^T \\
M &= [C_{20} \; C_{22} \; C_{21} \; S_{21} \; S_{22}]^T
\end{aligned}
$$

one can write the equations determining $\varkappa$, φ, ω as

$$f = f(e, M) = 0$$

and the solution will be of the form

$$e = e(M).$$

The respective variance-covariance matrices of M and e, Σ_M and Σ_e, are

$$\Sigma_e = \left[\frac{\partial e}{\partial M}\right] \Sigma_M \left[\frac{\partial e}{\partial M}\right]^T \qquad (C.9)$$

Taking the total differential of $f = f(e, M) = 0$

$$df = \left[\frac{\partial f}{\partial M}\right] dM + \left[\frac{\partial f}{\partial e}\right] de = 0$$

so that

$$de = \left[\frac{\partial f}{\partial e}\right]^{-1} \cdot \left[\frac{\partial f}{\partial M}\right] dM$$

or

$$\left[\frac{\partial e}{\partial M}\right] = -\left[\frac{\partial f}{\partial e}\right]^{-1} \left[\frac{\partial f}{\partial M}\right] \qquad (C.10)$$

The matrices $\left[\frac{\partial f}{\partial e}\right]\left[\frac{\partial f}{\partial M}\right]$ can be found from equations (C.3), after A - B, A - C, D, E and F are substituted with C_{20}, C_{22}, C_{20}, S_{21}, S_{22} by means of equations (C.5). However since the angles $\varkappa$, φ and ω are small, one can use equations (C.8) to obtain

$$\left[\frac{\partial f}{\partial e}\right] = \begin{bmatrix} -C_{21} & 2S_{22} & C_{20} + 2C_{22} \\ S_{21} & 2C_{22} - C_{20} & -2S_{22} \\ -4C_{22} & -S_{21} & C_{21} \end{bmatrix}$$

$$\left[\frac{\partial f}{\partial M}\right] = \begin{bmatrix} \omega & 2\omega & -\varkappa & 1 & 2\varphi \\ -\varphi & 2\varphi & 1 & \varkappa & -2\omega \\ 0 & -4\varkappa & \omega & -\varphi & 2 \end{bmatrix} . \tag{C.11}$$

After analytical inversion

$$\left[\frac{\partial f}{\partial e}\right]^{-1} = \frac{1}{d} \left[\begin{array}{c|c|c} 2S_{21}S_{22} - C_{21}(2C_{22} - C_{20}) & 2C_{21}S_{22} + S_{21}(C_{20} + 2C_{22}) & 4S_{22}^2 + 4C_{22}^2 - C_{20}^2 \\ C_{21}S_{21} - 8C_{22}S_{22} & C_{21}^2 - 4C_{22}(C_{20} + 2C_{22}) & 2C_{21}S_{22} - S_{21}(C_{20} + 2C_{22}) \\ S_{21}^2 - 4C_{22}(2C_{22} - C_{20}) & C_{21}S_{21} + 8C_{22}S_{22} & 2S_{21}S_{22} + C_{21}(2C_{22} - C_{20}) \end{array}\right] \tag{C.12}$$

where $d = 2C_{22}(2C_{20}^2 + C_{21}^2 - 8C_{22}^2 + S_{21}^2 - 8S_{22}^2) + C_{20}(S_{21}^2 - C_{21}^2)$.

With $\left[\frac{\partial f}{\partial M}\right]$ and $\left[\frac{\partial f}{\partial e}\right]^{-1}$ from (C.11) and (C.12), one can compute $\left[\frac{\partial e}{\partial M}\right]$ by means of (C.10) and finally with Σ_M, one can find Σ_e from (C.9).

As an example, to get an idea of the order of the magnitude involved, examine the simple case when $\varkappa = \varphi = \omega = 0$, $C_{21} = S_{21} = S_{22} = 0$, all the coefficients have the same variance s^2 without any correlation, and from [10], $C_{20} = -204.8 \times 10^{-6}$, $C_{22} = 22.1 \times 10^{-6}$.

For this case

$$\left[\frac{\partial f}{\partial e}\right] = \begin{bmatrix} 0 & 0 & C_{20} + 2C_{22} \\ 0 & 2C_{22} - C_{20} & 0 \\ -4C_{22} & 0 & 0 \end{bmatrix}.$$

$$\left[\frac{\partial f}{\partial e}\right]^{-1} = \begin{bmatrix} 0 & 0 & (-4C_{22})^{-1} \\ 0 & (2C_{22} - C_{20})^{-1} & 0 \\ (C_{20} + C_{22})^{-1} & 0 & 0 \end{bmatrix}$$

$$\approx 10^4 \begin{bmatrix} 0 & 0 & -1.13 \\ 0 & 0.40 & 0 \\ -0.62 & 0 & 0 \end{bmatrix}$$

$$\left[\frac{\partial f}{\partial M}\right] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix},$$

$$\left[\frac{\partial e}{\partial M}\right] \approx 10^4 \begin{bmatrix} 0 & 0 & 0 & 0 & 2.26 \\ 0 & 0 & -0.40 & 0 & 0 \\ 0 & 0 & 0 & 0.62 & 0 \end{bmatrix}$$

$$\Sigma_e \approx s^2\, 10^8 \begin{bmatrix} 5.11 & 0 & 0 \\ 0 & 0.16 & 0 \\ 0 & 0 & 0.39 \end{bmatrix} .$$

The variance of the angle $\varkappa$ is

$$\sigma^2_{\varkappa} = 2.26 \times 10^4\, s .$$

With the current accuracy of $s = 3.0 \times 10^{-6}$ [10]

$$\sigma^2_{\varkappa} = 6.78 \times 10^{-2} \text{ rad} = 118 \text{ km}/1738 \text{ km}$$

i.e., a standard deviation of about 120 km of arc on the lunar surface.

To obtain an accuracy of 1 km, potential coefficients would need to be

$$s = \frac{\sigma_{\varkappa}}{2.26 \times 10^4} = \frac{1 \text{ km } 10^{-4}}{1738 \text{ km } 2.26} = 2.5 \times 10^{-8}$$

i.e., an improvement of the current accuracy by a factor of 100 would be needed.